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ABSTRACT

This document is written for middle school teachers of grades five through nine who do not have specialized backgrounds in geometry. It is arranged in three parts. The first part provides a brief overview of the geometry curriculum of the middle school that includes the present state of affairs, a rationale for inclusion of geometry in the curriculum, the geometry that is suggested for instruction, and suggestions for teaching methods. Part two covers the following selected topics: axiomatic systems and models, distance, congruence, constructions, and transformational geometry. The material in this part is not designed for immediate use, but requires adaptation to particular classroom settings. Most sections include suggested exercises, learning activities, and selected references. The third part is an extensive bibliography of references for both readings and additional activities in geometry. (MP)

Geometric Selections for Middle School Teachers

(5-9)

by Douglas B. Aichele
and Melfried Olson

The Curriculum Series

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Note

The opinions expressed in this publication should not be construed as representing the policy or position of the National Education Association. Materials published as part of the NEA Curriculum Series are intended to be discussion documents for teachers who are concerned with specialized interests of the profession.

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TO KATHRYN
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The Authors

Douglas B. Aichele is a Professor and Head of the Department of Curriculum and Instruction, Oklahoma State University, Stillwater. Dr. Aichele is the editor of *Mathematics Teacher Education: Critical Issues and Trends* published by NEA.

Melfried Olson is with the Science and Mathematics Teaching Center, University of Wyoming, Laramie.

The Advisory Panel

Judy Adams, Mathematics teacher, Laramie Junior High School, Wyoming

John C. Arch, 6th grade teacher, Park Avenue School, Nashville, Tennessee

Helen Neely Cheek, Assistant Professor, Oklahoma State University, Stillwater

Genevieve M. Ebbert, 4th and 5th grade teacher, Lexington Public Schools, Massachusetts

Ray Kurtz, Professor of Curriculum and Instruction, Kansas State University, Manhattan

Lee E. Vochko, Algebra and Geometry teacher, Capistrano Valley High School, Mission Viejo, California

Contents

Introduction

This publication is written for middle school teachers of grades 5–9 without specialized backgrounds in geometry. We have arranged it in three parts, hoping that children and young adolescents will thus have an opportunity to experience some of the activities and suggestions which we have presented.

Part I is a brief overview of the geometry curriculum of the middle school including the present state of affairs, a rationale for inclusion of geometry in the curriculum, the geometry that we believe should be taught, and suggestions for teaching it.

Part II is a presentation of several selected topics for study: axiomatic systems and models, distance, congruence, constructions, and transformational geometry. This material is specifically not organized for immediate classroom use, but rather requires adaptation to the particular classroom setting. Most of the sections include thought-provoking supportive exercises worthy of discussion, activities that are ready for immediate classroom use, and references for further study. We purposely omitted all "proofs" from the written presentations, though many of the exercises provide opportunities for this activity.

Part III is an extensive collection of references for additional activities as well as readings in the area of geometry. We hope this bibliography will assist teachers in enriching their own geometry curriculum and in developing their own activities.

Douglas B. Aichele
Melfried Olsen

An Overview and Commentary

PRESENT STATE OF AFFAIRS

The word "geometry" is from the Greek word "geometrein," which is composed of the parts "geo" (earth) and "metrein" (to measure), since geometry was originally the science of measuring the land. It has a rich and beautiful historical development which grew primarily out of the human quest to understand and quantify the environment. In geometry, as in other subjects, the emphasis on the pragmatic too often overrides the aesthetic in our instruction. Although geometry is a very practical subject, the history of the development of geometry is as interesting as virtually any other voyage through time.

The most complete attempt to assess individuals' geometric knowledge was undertaken by the Education Commission of the States through the National Assessment of Educational Progress. The following two excerpts provide information about what students know and can do.

> Most of the geometry items given to 9-year-olds involved identification of terms. Nearly all could name a circle and triangle, but only about one-fourth could identify a cone, sphere, or ellipse. Almost half of the children could select a pair of parallel lines from a set of several different configurations of line pairs, but only 20% of them successfully drew a line parallel to a given line. About one-third of the children could select a right angle from the alternatives of right, acute, or obtuse angles (4, p. 4).

> Some geometric figures are recognized by most 17-year-olds, but they do not know many other geometric facts. Nearly all the students named a circle, 75% named a cone, and about 50% named a cylinder. Fewer than one-half named either a cube or a sphere. About 75% knew that the diameter of a circle is twice the length of a radius. Fewer than 40% could bisect an angle using a straightedge and a compass. Slightly more than one-fourth of the 17-year-olds found the area of a square with a perimeter of 12 inches (4, p. 81).

These statements provide us with indications that some geometry is taught to and learned by students, but they also indicate that improvements can be made concerning the teaching and learning of geometry. The National Council of Teachers of Mathematics conducted a study in 1975 of second and fifth grade classrooms and found that "though geometry is mentioned as being part of texts, objectives, and testing, 78% of the teachers report spending fewer than 15 class periods per year on geometric topics" (1, p. 13).

Thus, we find that while geometry is considered a viable part of the curriculum, most teachers seem to pay only lip service to that fact and instead concentrate on teaching number concepts and arithmetic skills. We suspect that teachers' lack of knowledge and previous success in geometry account for their position in this regard. It is any wonder, then, that children's geometric knowledge is weak?

In order to get full benefit from Part II of this book, teachers must have some background in the basic concepts undergirding the subject of geometry. We certainly do not advocate the teaching of geometry to pupils in the middle school in a manner presented here—that is why we have provided the extensive bibliographical background and the selected activities. We hope that readers will feel convinced that geometry should be taught and will not hesitate to implement their convictions. With the commonly held notion that we teach as we are taught, we hope that teachers and teachers-to-be can use the presentations found herein as a model for future presentations on similar topics.

WHY GEOMETRY?

The past twenty years have demonstrated that students are concerned with the exploration, study, and measurement of space. Geometric ideas such as position, direction, size, shape, and similarity are inherent in a child's earliest experiences. In fact, some conjecture that by the time children enter school, their intuitive knowledge of geometry is as great as their developmental concepts of number. Almost every object has inherent geometric properties, and as children internalize these properties they are primed for future activities built around cubes, prisms, cones and, later, rectangles, squares, and triangles. The use of geometry in many professions, including art, engineering, and architectural designs, is undeniable. Geometry is all around us! This may be an oversimplification, but we live in a geometric world—open your eyes on your next excursion through a day of your life.

> Geometry is an indispensable way of life; for in its very structure, form, and beauty, every imaginable object is governed by properties of geometry. Without this tool of knowledge, our very lives, our needs, and the world about us would be altered beyond any describable form. Yet, because of its inherent structure and utility, geometry finds its application in the vocational needs of the most demanding professions: architecture, engineering, navigation, physics, and art, to mention a few. For this reason, many people feel it is too technical a subject for them to understand, and certainly beyond young children. However, in daily life we can identify innumerable instances where geometric insights have been applied to the nearest objects. Geometry has been used to design the form of this page, the pencil used to write this article, the desk on which I am writing, the light bulb that illuminates the room, and the room itself, which is one of many in a building located at a point on this spherical planet (7, pp. 454–55).

All too often students remember geometry as the subject in which they "proved" theorems, although when pressed, they cannot recall what it was they proved, nor necessarily remember the structure of the proof of a particular theorem.

Geometry is not thought about in the same fashion as arithmetic, although there is much agreement on the fact that both subjects are important. With arithmetic, almost all people concur on the relevance of teaching addition, subtraction, multiplication, and division, but with geometry it's another matter. There are disparate views on how much, why, when, and to whom geometry should be taught. Despite this lack of consensus, geometry is on the rise in the elementary curriculum, and the enthusiastic teacher should be prepared to teach it, whatever it may be. The following quotations further emphasize this position:

> Elementary and upper grade teachers should consider giving geometry a prominent place in their mathematics programs because of the wealth of problem-solving activities with geometric content. Furthermore, the omission of geometry from a mathematics program will limit a teacher's opportunity to develop a student's non-numerical, mathematical ability (3, p. 444).

> The teaching of geometry in the elementary school can also be justified by the fact that children have an innate curiosity for the subject stimulated by environmental experiences before entering school. The students can see position, shape, and size in space as entities that can be used, controlled, and manipulated to explore their environment. And since geometry is a way of modeling our physical environment and because there is a great abundance of

models suitable for all levels, from kindergarten through graduate school, geometry is a natural vehicle for developing intuition, creativity, and a spirit of inquiry. Further, geometry is a fertile source for interesting and challenging problems, and geometric methods are powerful tools in problem solving (6, p. 469).

Jean Piaget provides us with the clearest evidence that school children have the ability to learn geometric concepts. His research indicates that children's initial understanding is of simple topological concepts (closure, interior, exterior, continuity) prior to progressing through projective and then formalized Euclidean concepts with topological properties providing the foundation upon which the rest are formulated. Furthermore, similar research supports the conclusion that children can learn geometry at an early age without damaging conceptual development of other concepts (i.e., arithmetic). The development learning advocated by Piaget has implications here also. If children learn by manipulating their environment, then geometry will be useful for this manipulation. As noted earlier, the environment of a child is filled with geometry, including topological, projective, and Euclidean concepts with intuitive three-dimensional conceptual development occurring very early and prior to two-dimensional development. The adept teacher will not only provide geometric solids for the children to observe, manipulate, and measure, but will also include spatial experimentations similar to those provided by the use of ESS geoblock activities. Not only will this provide an effective technique for teaching, but it provides a rationale for the concepts taught.

Teachers need to be cognizant of these considerations to understand what will happen in the classroom. Children, as should be expected, will be inaccurate in their initial attempts at geometry at this early age. Teachers should, however, encourage the child to set out for himself or herself and not worry about early mistakes that can be corrected later. To continually refresh us in our thoughts about "Why Geometry?" we should consider the message found in the following two citations:

> In an effort to save our own and the children's time, we often conclude that the learner does not need firsthand experience. We feel justified and sometimes even benevolent or noble in imparting to tender young minds the formula for a triangle's area or for the volume of a pyramid. In so doing we usurp the rights of our children, and we deceive ourselves. We usurp the rights of children if we tell a child the secret of some truth or principle, because we drain from him the zest for finding it out. We remove from him the need for inventiveness. We remove his opportunity to face a challenge from which he might learn greater trust in himself and greater skill in thinking. We remove from him some excitement about learning.
>
> We deceive ourselves when we equate telling with teaching. To be told and to be taught are not the same thing. To be told is to be released from obligation. To be told is to become other-dependent, rather than self-dependent. To be told is to be denied the growth-producing experience of reaching out to capture another truth.
>
> Telling too much to children is not a fault of teachers alone. Some textbooks also tell far too much. Occasionally a teacher feels obligated by a text to deal with a topic in a manner that is inconsistent with his/her beliefs about how children best learn (5, p. 85).

One can hardly speak of effective instruction in geometry without recalling remarks made by Werthcimer. He reminded us that children who are taught specific solutions or techniques cannot handle variations, because they fail to

react to the inner relations of the problem. They fail to see the relationship of the parts to the whole. A learner should be aware of the structural features of a problem that set up strains, stresses, and tensions. Structural features create vectors, or mental forces, and determine their direction, quality, and intensity. These in turn lead to the steps and operations that fulfill the requirements of the problem (5, p. 86).

WHAT GEOMETRY SHOULD THE CURRICULUM CONTAIN? SUGGESTIONS FOR TEACHING

There is growing evidence among mathematics educators that geometry should be experienced in each year of schooling from kindergarten through grade 12. Geometry is the study of spatial relationships of all kinds, relationships that can be found in the three-dimensional space we live in and on any two-dimensional surface in this three-dimensional space. These relationships can be discovered all around us. Observe the many different shapes in your environment. This is geometry. Listen to the description of the path of the latest space rocket. This is geometry. Compare the photograph taken with a polaroid camera to the object that it pictures. This is geometry. Notice the symmetry to be found in a spherical or cubical shape and the lack of symmetry in some modern works of sculpture. This is geometry. All of these involve spatial relationships. Children are aware of spatial relationships from their earliest days. Introducing them to the idea of geometry as being concerned with shape and size in the material world will help them to realize and appreciate that mathematics is something that plays an important role in the world in which we live (2, p. 473).

The emphasis placed on geometry by the mathematics educator is not new, as similar feelings were exhibited in textbooks at the turn of the century. Furthermore, geometry was the classical study by the early mathematician from which all other aspects of mathematics were initially derived. In fact, educators today feel that the study of geometry, equality, congruence, similarity, symmetry, etc., provides a rich source for the visualization of arithmetic and algebraic models. The explorative study of two- and three-dimensional figures and the transformations of those figures in a child's environment reflect an introduction to significant sophisticated mathematical ideas which the child can investigate prior to the formal introduction via precise definitions and theorems.

The intuitive introduction at any early age, plus the solid geometry and transformation topics which follow at a later age, are useful for future scientific work. However, many parents, and especially teachers, promote a nonenthusiastic response to this situation. Geometry has often degenerated into an effort to prove theorems and has thus derived a present definition that may be difficult to change. These disagreements, the slow acceptance, and disagreements concerning inclusion of the new topics demonstrate the lack of consensus over the appropriate purpose and worth of geometry in the curriculum.

Having established objectives, the question of what geometry should be taught is, however, still not answered. Over the past ten or fifteen years there have been the proponents of synthetic geometry, of analytic geometry, of vector geometry, and of transformational geometry. Which of these geometries should we teach elementary teachers? The answer is, "None of them,

and all of them." This response is not as ambiguous as it sounds. With reference to "none," no one geometry should comprise an entire course. It is a waste of time to put elementary teachers through a careful development of any single geometry. The advocates of a careful development of one-geometry support their position with the contention that geometry is an excellent vehicle for teaching the nature of a mathematical system. We're kidding ourselves if we think that this is what we're accomplishing in teaching geometry to all except the most sophisticated students (9, p. 458).

We believe that some of the techniques and strategies particular to each of the geometries will prove beneficial as problem-solving tools. For example, many difficult problems posed in a synthetic setting are resolved easily by employing an analytic argument. Hoewever, students typically are not able to solve problems by considering such alternative strategies. They merely approach the problem with the "given" information and try to create a string of statements which evolves into what is to be "proved."

We envision a geometry experience for middle school teachers which not only includes topics and techniques selected from synthetic, analytic, transformational, and vector geometry, but also provides opportunities to examine three-dimensional space concepts. Furthermore, teachers should have the opportunity to examine typical Euclidean concepts in other settings (e.g., distance, congruence). We make a plea for study of alternative distance definitions such as the "taxicab" distance as well as for an examination of alternative models such as the sphere and certain finite systems. In times when technology has advanced to the point where the earth is readily perceived in a spherical sense, we cannot ignore, nor expect our children to ignore, the geometry associated with this historically famous model. Furthermore, there is probably no more intuitive model available for our scrutiny.

We close with a few suggestions that teachers can use in teaching informal geometry. First, design and implement activities that require pupils to manipulate, trace, or construct geometric figures and solids. Classification by comparison is a natural sequel to discovering properties of these figures and solids. Next, design activities aimed at discovery relationships (e.g., shape, size) among the figures or solids. Finally, plan activities focusing on sliding, flipping, and turning these figures or solids. For example, examining whether the size and/or shape of a right triangle changes when it is rotated is as important a consideration as examining whether the number property associated with a group of checkers changes when the checkers are placed in a circular arrangement instead of a linear arrangement.

You will note that these suggestions in no way excluded the sphere, though most readers probably will not interpret them in this way. Why not? Because our frame of reference is typically Euclidean. May we suggest you try a unit on informal geometry on the sphere (study Part II, Section 2.1.2) first.

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2.1

Axiomatic Systems and Models

2.1.1 WHAT IS AN AXIOMATIC SYSTEM?

The Stillwater Racquetball Club is sponsoring a tournament. The major concern in organizing the tournament consist of making all the necessary preliminary arrangements, providing a hospitality suite during the tournament, coordinating the use of the court facilities during the tournament, and providing the tournament winners with trophies. Clearly some organizational scheme is necessary to enable the club to host the tournament effectively. A committee approach seemed the best strategy, with three basic rules for forming the committees:

Rule 1. Each pair of committees has exactly one member in common.

Rule 2. Each committee member is on exactly two committees.

Rule 3. There are exactly four committees.

Le.'s see now. Basing your reasoning on these rules, how many club members will be serving on the committees? How many members will each committee have? At least two interpretations of these rules become apparent. One is:

ABC

ADE

BDF

CEF

Figure 2.1a

Here single letters represent "committee members" and rectangles represent "committees." So, for example, "A" is a committee member and "BDF" is a committee.

Another interpretation has a more geometric orientation: In Figure 2.1b "points" and "lines" replace "members" and "committees." That is, "A" is realized as a point and "BDF," a line.

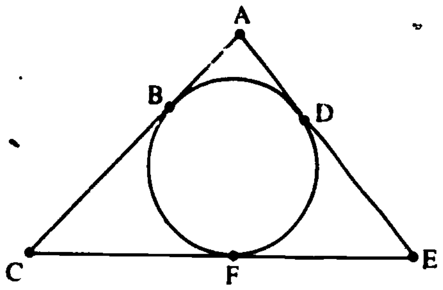

Figure 2.1b

Regardless of which interpretation of the rules is being made, a total of six club members will be committee members and each committee will have exactly three members. How does one know there is not another interpretation of these rules with more than six members?

This rather simple example illustrates much of the background necessary for studying axiomatic systems. First of all, it is desirable for individuals who are engaging in beneficial discourse to agree on most of the words and terms used, though frequently this is the reason for the discussion! At any rate, it is very difficult to offer precise, universally accepted definitions for many of our words. In some ways mathematicians recognize this dilemma and through the very structure of mathematics offer an alternative, which is to "accept without definition" some of these words and concepts. We refer to such entities as *primitive* or *undefined terms*. What are the primitive or undefined terms in the introductory example?

If we agree that certain terms will not be defined, then we probably should also accept that certain statements involving these undefined terms will be accepted without verification. We refer to such statements as *axioms* or *postulates*. One might wonder just how many of these undefined terms are permissible in a mathematical system. As a matter of convenience and economy it is desirable to hold this number to a minimum. In our introductory example, the axioms were the three rules concerning committee membership. These rules involved the notions of "committee" and "committee member," which were the primitive or undefined terms.

When we asked the question, "How many club members are on each committee?", we were really addressing the consequences or implications of the rules. In other words, certain statements "follow logically" from the rules and one another and provide additional information about the system. We must agree on certain rules of reasoning so that we will be able to determine how and when one statement "follows logically" from another. These logical consequences are called *theorems*.

We must remark here that while some concepts are left undefined, e.g., the axioms, many are not. These statements, called *definitions*, must also respect the laws of logical reasoning which undergird the mathematical system. For instance, in the introductory example, if we were to define a subcommittee as a subset of a committee with exactly two members and define each subset of a committee with two members as a subcommittee, we would be employing circular reasoning, thus violating a premise of logical reasoning.

Now, we are ready to synthesize our findings and more formally characterize an *axiomatic* or *postulational system* consisting of the following:

1. A set of undefined terms which form the basis of the necessary vocabulary.
2. A set of axioms or postulates.

3. Laws of formal reasoning, e.g., laws of Aristotelian logic.
4. A set of theorems which present properties of the undefined terms and which are derived from the axioms by the laws of logic.
5. A set of definitions.

On the surface, the construction of a postulational system may appear quite trivial. After all, one has only to accept some undefined terms and axioms and then apply a few well known principles of logic to derive additional statements—nothing to it! Furthermore, after this activity has been completed, one cannot really say much about its merit as it was all based on undefined premises!

This is all true, but there is a little more to it. If a person was moved to develop a postulational system, that person would undoubtedly be concerned about illustrating or realizing it in some more or less intuitive way. Such an illustration serves as a model of the system. More formally, a *model* of a postulational system exists if each of the undefined terms has been assigned some meaning so that all of the consequences are true.

To illustrate the relationship between a postulational system and a model of that system, recall the introductory example. The postulational system consisted of:

1. two undefined terms—"committee" and "committee member"
2. three axioms—Rules 1, 2, and 3
3. definitions and theorems which respect the laws of formal logic and which would be included as the system was enlarged.

A model for that system was illustrated in Figure 2.1a where the committee members are the elements of the set $\{A, B, C, D, E, F\}$ and the committees are enclosed in the rectangles; that is, the committees are designated ABC, ADE, BDF, and CEF. If this interpretation is in fact a model for the postulational system, then each of the undefined terms must be assigned some meaning (have they been?) and each of the consequences must be true under this assignment. Let's examine this last requirement more closely.

First, is Rule 1 true in this model? That is, does each pair of committees have exactly one member in common? Yes, we can see this by examining the table below which shows, for example, that committees ABC and ADE have only member A in common.

Committees

Committees	ABC	ADE	BDF	CEF
ABC	—	A	B	C
ADE	A	—	D	E
BDF	B	D	—	F
CEF	C	E	F	—

Does Rule 2 hold? That is, is each committee member on exactly two committees? Yes, we can also see this by examining the table below which shows, for example, that member B is on only committees ABC and BDF.

Committee Members

Committees	A	B	C	D	E	F
	ABC	ABC	ABC	ADE	ADE	BDF
	ADE	BDF	CEF	BDF	CEF	CEF

Now, what about Rule 3? Yes, it holds since our model was exactly four committees—ABC, ADE, BDF and CEF.

So, interpreting committee members by the letters A, B, C, D, E, F and committees by ABC, ADE, BDF, and CEF qualifies it as a model of the given postulational system.

Before leaving this model, we note that since there are six (finite number) committee members, we refer to this model as a *finite model*. Had it been the case that an infinite number of committee members was possible, we would have referred to it as a *infinite model*.

Now, let's consider another postulational system and a model for it. The undefined terms are: a set **C** of elements called *coins;* a set **B** whose elements are subsets of **C** called *bins*. This system has five postulates:

Postulate 1: There exists at least one bin.

Postulate 2: There are exactly three coins on every bin.

Postulate 3: Not all coins are on the same bin.

Postulate 4: There is exactly one bin on any two distinct coins.

Postulate 5: There is at least one coin on any two distinct bins.

We can exhibit a model for this system by letting $\mathbf{C} = \{A, B, C, D, E, F, G\}$ and **B** be comprised of those subsets of **C** which form the columns of the array below:

A	B	C	D	E	F	G
B	C	D	E	F	G	A
D	E	F	G	A	B	C

We must show that each of the postulates has true meaning when interpreted in this way. That is, we must examine each postulate very carefully to be sure that the interpretation being made is true. We will not verify all of the postulates here as this is the content of Exercise 1 at the end of the section. However, as an example, let's examine the interpretation of Postulate 4. We must be able to select any two distinct coins and exhibit exactly one bin containing them. Perhaps this is most easily accomplished by scrutinizing the table below which was prepared by considering all of the possible pairs of coins and the bins containing them. The entries in the table are exhaustive and clearly show that each pair of distinct coins is contained on exactly one bin.

Coins	*Bin*	*Coins*	*Bin*	*Coins*	*Bin*
A, B	ABD	B, D	ABD	C, G	GAC
A, C	GAC	B, E	BCE	D, E	DEG
A, D	ABD	B, F	FGB	D, F	CDF
A, E	EFA	B, G	FGB	D, G	DEG
A, F	EF.	C, D	CDF	E, F	EFA
A, G	GAC	C, E	BCE	E, G	DEG
B, C	BCE	C, F	CDF	F, G	FGB

If we replaced the undefined terms "coin" with "point" and "bin" with "line" we would have a finite postulational system usually referred to as *Fano's 7-Point Geometry*. This system could be modeled by letting the set of **points** be $P = \{P_1, P_2, P_3, P_4, P_5, P_6,$

$P_7\}$ and the set of **lines** be comprised of these subsets of P which form the columns of the array below. That is, P_1 is a "point" and $\{P_1, P_2, P_4\}$, or simply $P_1P_2P_4$, is a "line."

P_1	P_2	P_3	P_4	P_5	P_6	P_7
P_2	P_3	P_4	P_5	P_6	P_7	P_1
P_4	P_5	P_6	P_7	P_1	P_2	P_3

Many of the notions we have come to use are derived through intuition, guessing, trial and error, and a host of other less than systematic means. Other than the fact that the axiomatic approach addresses the very heart of the structure of mathematics, it is the means by which we can *prove* that certain statements are correct. Children and adolescents should have opportunities aimed at developing skills involving intuition, guessing, estimating, etc. However, they should also have opportunities to make conjectures and verify the accuracy of these conjectures, as well as the conjectures of others, within a mathematical framework. Several postulational systems together with models of them are given in the exercises to provide such opportunities.

You will note that most of the examples given above make use of combinatoric logic. The use of combinatoric logic by students on a task enables us to classify them as "formal thinkers" respective to that task. It is during the middle school years that students are making the transition from concrete to formal thought. Give them the opportunity to see how they approach problems of this type.

Exercises

1. Verify that the "coins/bins" interpretation of the 7-Point Geometry is a model for that system.

For the remaining exercises, it is understood that the undefined terms are "point" and "line."

2. 13-Point/13-Line Geometry

   Postulate 1: There exists at least one line.
   Postulate 2: There are exactly four points on every line.
   Postulate 3: Not all points are on the same line.
   Postulate 4: There is exactly one line on any two distinct points.
   Postulate 5: There is at least one point on any two distinct lines.

   Let the set of points $P = \{P_1, P_2, P_3, \ldots, P_{13}\}$ and the set of lines be comprised of those subsets of P which form the columns of the array below:

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}
P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_1
P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_1	P_2	P_3
P_{10}	P_{11}	P_{12}	P_{13}	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9

   Show that this interpretation is a model of the postulational system.

3. Young's 9-point/12-line geometry

Postulate 1: There exists at least one line.
Postulate 2: There are exactly three points on every line.
Postulate 3: Not all points are on the same line.
Postulate 4: There is exactly one line on any two distinct points.
Postulate 5: Given a line l and a point P not on l, there is exactly one line on P and not on any point of l.

Let the set of points $P = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9\}$ and the set of lines be comprised of those subsets of P which form the columns of the array below:

P_1	P_1	P_1	P_1	P_2	P_2	P_2	P_3	P_3	P_3	P_4	P_7
P_2	P_4	P_5	P_6	P_5	P_4	P_6	P_4	P_5	P_6	P_5	P_8
P_3	P_7	P_9	P_8	P_8	P_9	P_7	P_8	P_7	P_9	P_6	P_9

Show that this interpretation is a model of the postulational system.

4. Affine Geometry

Postulate 1: There exists at least one line.
Postulate 2: There are at least two points on every line.
Postulate 3: Not all points are on the same line.
Postulate 4: There is exactly one line on any two distinct points.
Postulate 5: Given a line l and a point P not on l, there is exactly one line on P and not on any point of l.

Let the set of points be the 25 letters A, B, C, ..., Y (all the letters of the English alphabet except Z). The lines are the 30 sets of five letters which occur together in any row or any column of the three blocks below:

A	B	C	D	E		A	I	L	T	W		A	X	Q	O	H
F	G	H	I	J		S	V	E	H	K		R	K	I	B	Y
K	L	M	N	O		G	O	R	U	D		J	C	U	S	L
P	Q	R	S	T		Y	C	F	N	Q		V	T	M	F	D
U	V	W	X	Y		M	P	X	B	J		N	G	E	W	P

Show that this interpretation is a model of the postulational system.

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2.1.2 TWO FAMILIAR MODELS

The intent of this section is really to examine two models of a well-known postulational system. But before considering these models, we must be introduced to the postulational system.

The undefined terms are "point" and "line." We define "plane" to be the set of all points under consideration.

The initial postulates establish relationships among the undefined terms. They are:

Postulate 1: There exist at least two lines.

Postulate 2: Each line is a set of points containing at least two elements.

Since postulational systems are developmental by nature, one must be prepared to make decisions that will affect the direction that the system takes. For example, we are now in a position to decide whether or not to introduce the concept of distance into the system. It is advantageous for us to infuse this concept for at least two reasons. First, we will be better able to explore relationships between number concepts and geometric concepts. Also, this is the approach most closely followed in school geometry classes. Since the distance concept is so pervasive in the geometry curriculum, a special section has been prepared (Section 2.2) which not only addresses distance in our evolving postulational system but interprets this concept in several models.

Postulate 3: Corresponding to each pair of points P and Q there exists a nonnegative real number PQ satisfying the conditions: (i) PQ = 0 if and only if P = Q and (ii) PQ = QP.

We define the nonnegative real number corresponding to the points given in Postulate 3 as the *distance between P and Q*. The postulate and definition appear intuitively reasonable since distance between points is positive except when the two points are the same point (in this case, the distance is understood to be zero). Also, the wording of the definition suggests that distance is not directional. Furthermore, it is important to observe that the postulate assigns a nonnegative real number to *every* pair of points.

A very timely question concerns the *maximum* distance between any arbitrary pair of points. Let's investigate two models of this evolving postulational system and address the issue of maximal distance between an arbitrary pair of points in each model.

The first model and clearly the most well-known is attributed to Euclid, a Greek mathematician, around 300 B.C.. Euclid desperately attempted to provide definitions for all terms. He defined a "straight line" as "that which lies evenly with the points on itself," and a "point" as "that which has no part." Neither of these definitions proves to be very useful or informative; we really can regard them only as undefined terms. It is also appropriate to regard "line" and "straight line" as synonymous. The notion of a "plane" is interpreted as the "flat surface" which contains all of the points and lines.

The notion of Euclidean distance is addressed in Section 2.2.3. For our purposes in this section, however, we need only remember that Euclidean distance is best understood as the length of the path "as the crow flies." We also acknowledge that two points can be arbitrarily far apart and hence there is no real number which is the maximum distance that points can be apart. So, in the Euclidean model, distances are not bounded.

Now, let's turn our attention to a model which, with increasing technological advances and space exploration, becomes more intuitive than ever before. Imagine that the plane is the surface of a sphere—think of the plane as the surface of a globe, basketball, or beachball. Interpret points as in the Euclidean model and lines as great circles of the sphere. Recall from your studies in geography, that a great circle is determined by intersecting a plane containing the center of the sphere with the sphere itself. The concept of plane, point and line is illustrated in Figure 2.1c.

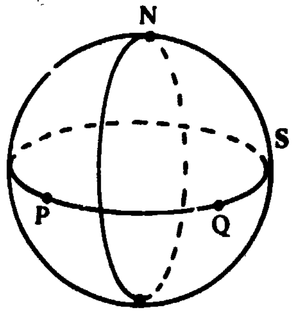


Figure 2.1c

The distance between two points on the sphere is defined to be the minimal length along great circle paths having those two points as endpoints. This is shown in Figure 2.1d where the minimal great circle path is denoted d_s (A, B). This interpretation seems rather intuitive except when the points are "poles"—like the north pole N and south pole S in Figure 2.1c. This distance, d_s (N, S), is one-half the circumference of any great circle and the maximum distance between any two points on the surface of the sphere (why?). This means that we can determine the maximum distance between any two points on a sphere unlike the situation which prevailed in the Euclidean model. Thus, we say that distance is *bounded* on the sphere.

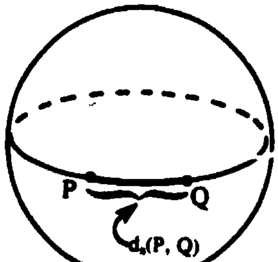


Figure 2.1d

We close this section with an analysis of a familiar statement.

Two distinct points determine only one line.

Is this statement "true" or "false"? Most people would say the statement is true as they recall their studies of Euclidean geometry. But for those who interpret this statement in the spherical model, this conclusion is not so immediate. Consider the north pole N and the south pole S in Figure 2.1e. It is clear that there is not a unique line passing through N and S, but instead there are *infinitely* many lines passing through N and S—lines of longitude. So, the original statement is "false" when interpreted in the spherical model. The heart of the problem with this statement seems to reside in the fact that the distance between N and S is the maximal distance.

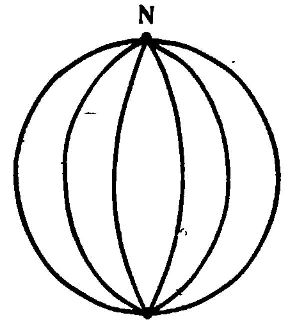


Figure 2.1e

Regardless of the model, the statement that "two distinct points lie on at least one line" is correct. It could be strengthened to include a condition of uniqueness by disallowing the points to be the maximum distance apart. So,

> Two distinct points lie on at least one line, and if the distance between the points is not the maximum distance, the line is unique.

is a statement with true interpretations in both the Euclidean and spherical models.

We believe it is critical for students to be allowed to evaluate statements within a given context, i.e., within a given model. The spherical model, being so intuitive and available, provides a natural opportunity for such activities. Several typically Euclidean statements are included in Exercises 4–10 with this intent, and many of these statements, as well as many statements found in the Activities, are addressed in detail elsewhere in this book.

Exercises

1. Verify the statement below as a consequence of only Postulates 1 and 2.

   There are at least three points and two lines.

2. Exhibit a model for the postulational system which includes only postulates 1 and 2. (Hint: let $P = \{P_1, P_2, P_3\}$ where P_1, P_2, P_3 represent points and L be the subsets $\{P_1, P_2\}$ and $\{P_1, P_3\}$ which represent lines).

3. Considering the earth as a sphere, name several of its great circles.

In the remaining exercises, familiar Euclidean statements are given. Examine each of them in the spherical model and decide if they are correct.

4. Given any three collinear points, one of them always falls between the other two.

5. Side-Angle-Side (SAS) Triangle Congruence Criteria.

6. The measure of any exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles.

7. Through a given point not on a given line there can be drawn exactly one line parallel to the given line.

8. The sum of the measures of the angles of a triangle is 180°.

9. Through three noncollinear points there is a circle.

10. The square of the hypotenuse of a right triangle equals the sum of the squares of the legs.

Activities

Examine the following concepts in the spherical model and decide how they differ from their usual Euclidean interpretations:

A. Point B is *between* points A and C

B. A *coordinate* system for a line

C. *Segment* joining points A and B

D. *Ray* beginning at A and containing B

E. *Angle* and *angle measurement*

F. *Betweenness* for rays with the same vertex

G. *Plane separation*

H. *Convex* set

I. *Congruence* of segments, angles and triangles

J. *Perpendicularity*

K. *Circles*

L. *Parallelism*

2.2 Distance

Distance! The word probably conjures up a clear meaning in the mind of the reader, but we conjecture that the meaning has been derived from a limited exposure to what distance can mean and how distance can be defined. It is the purpose of this section to provide an alternative look at the usual definition of distance, with some resulting ramifications, and to consider a definition of distance in the spherical model.

2.2.1 WHAT IS DISTANCE?

Well, first of all, distance results from a method of measuring, and this measuring is usually between two points. What this measuring does is assign a number (and unit) to two points, say six miles, to represent "how far apart" they are. Intuitively, we usually regard distance between points as positive, except when the points are the same, in which case the distance is understood to be zero. We also regard distances as *non*-directional, that is, the distance *from point P to point Q* is the same as *from Q to P*. Many students tend to want to associate direction with distance and, not surprisingly, this concept is considered when dealing with the introduction of integers. Thus, to avoid the problem, it is preferable to use the phrase "distance *between P and Q*" rather than "distance from P to Q." We shall use the preferred terminology.

Formally stated, we introduce the following two considerations.

Postulate. Corresponding to each pair of points P and Q there exists a nonnegative real number PQ satisfying the conditions: (i) PQ = 0 if and only if P = Q, and (ii) PQ = QP.

Definition. The nonnegative real number, PQ, corresponding to the points given in the above postulate is called the *distance between P and Q*.

With the above postulate and definition in hand, we shall proceed to look at two different models and three different ways of defining how we are going to measure distance.

2.2.2 SPHERICAL DISTANCE

We looked earlier at the spherical model where points had the usual connotation and "lines" are considered as great circles on the sphere. For our purposes, we define the distance between two points P and Q on a sphere as the *minimal* length along great circle paths having those two points as endpoints. If P and Q are not "poles" of a sphere, we see why we say minimal length, for consider Figures 2.2a and 2.2b where both paths have endpoints P and Q. The length of the path in Figure 2.2a is shorter and we choose its length to be the distance between P and Q, and denote it $d_s(P, Q)$.

What is the distance between P and Q if they are "poles?" In Figure 2.2c, if P and Q are the north and south poles, respectively, of a sphere with radius r, we note that the distance $d_s(P, Q)$ is one-half the circumference of any great circle (and if you recall the formula for circumference, $d_s(P, Q) = \frac{1}{2}(2\pi r) = \pi r$).

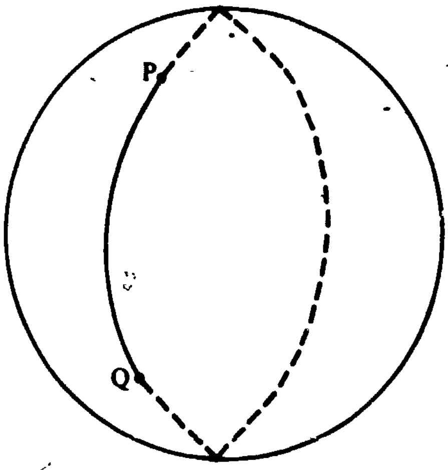


Figure 2.2a

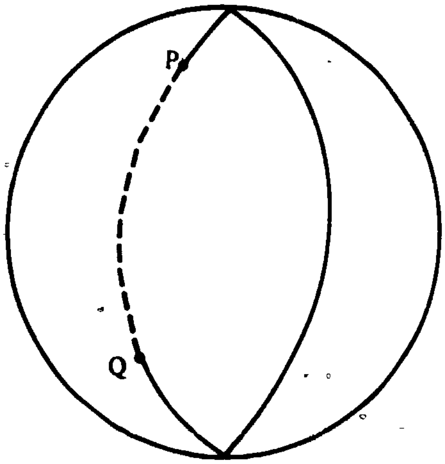


Figure 2.2b

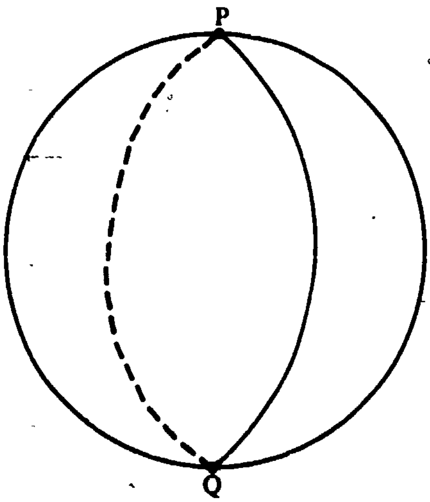


Figure 2.2c

We also note that the *maximum distance* between two points on a sphere with radius r is the distance between "poles," πr. Because of this, we have a situation on a sphere that does not occur in our other models—we have what we term a *finite* or *bounded* distance. That is to say, there is a *maximum distance* between any two points on a sphere.

We are familiar with the use of a number line, but how do we use this idea to measure distances on a sphere? Suppose we had a sphere whose maximum distance was M. If we cut off the usual number line at M and –M and bend it to make a circle (Figure 2.2d), we have a coordinatized circle which we can use to measure distances. The only problem we face is that when we connect the segment to make the circle, we have two names "–M" and "+M" for a single point. We resolve this situation by assigning "+M" as the coordinate of this point.

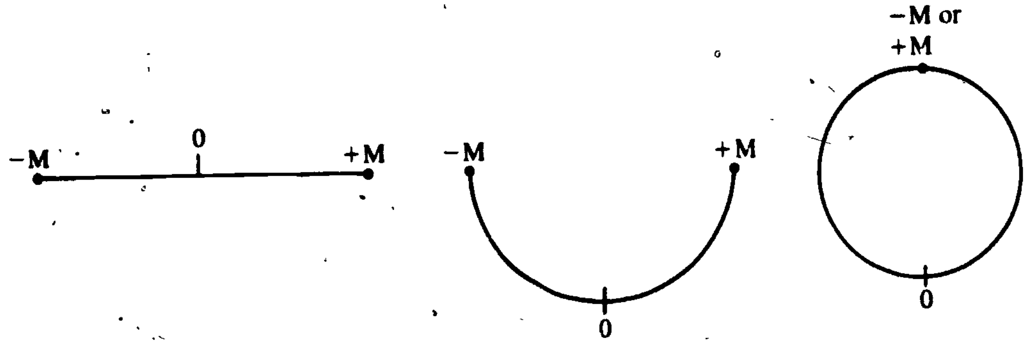


Figure 2.2d

Exercises

1. When P and Q are "poles," how many lines do contain P and Q?

2. If P and Q are distinct points, but not "poles," how many lines contain them?

3. Obtain a large spherical ball (soccer ball, basketball or beach ball) or a globe and mark two points on it that are not poles.
   (a) Using a long piece of string or wire, find the line that contains P and Q.
   (b) Using the line you found in (a), find the circumference of the ball. Using a meter stick, measure this length.
   (c) Using the line you found in (a), find the maximum distance between points on the ball.
   (d) Using the line you found in (a), find the distance between P and Q. Measure this distance with a meter stick.

4. Repeat Exercise 2 by picking different locations for points P and Q, until you are comfortable with the concept of "line containing P and Q," and the "distance between P and Q."

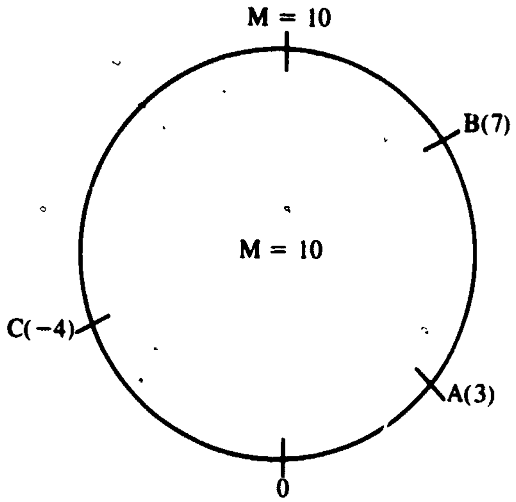
5. While it is clear that many of the concepts of spherical geometry can be illustrated using a ball of some sort, when it comes to working directly with great circles (lines of the sphere), the "basketball" illustration leaves something to be desired. Instead, a two-dimensional representation is preferred. A *circular geoboard* is exactly what we need. (Instructions for constructing a circular geoboard are given in the activity at the end of this section).

When considering the exercises which follow, we ask you to remember that children have had many experiences with a "circular geoboard" by the time they reach kindergarten, and they have mastered some skills by the end of the first grade. How? They have usually been exposed to a circular clock! Make sure you relate this exercise back to those experiences.

(a) Suppose we have a sphere where the maximum distance is 10, i.e., M = 10, and

A, B and C are points with coordinates 3, 7, −4, respectively, denoted A(3), B(7), C(−4) on the figure below.

Find $d_s(A, B)$, $d_s(B, C)$ and $d_s(A, C)$.



(b) Suppose M = 12 in (a) above. Find $d_s(A, B)$, $d_s(B, C)$, $d_s(A, C)$.

(c) Note $d_s(P, Q)$ sometimes depends upon M. For instance, $d_s(A, B) = 4$ in both 5(a) and 5(b), but $d_s(B, C) = 9$ in 5(a) while $d_s(B, C) = 11$ in 5(b).

Note: Some readers may recognize Exercise 2 from studies they previously performed in a history or social studies unit. The concept involved is "great circle routes." The above method will work well to show students that the shortest distance between two cities on a globe is found by considering a great circle route between those two cities. That this distance looks strange when transposed to a "flat" map, where students draw the "other straight line" with a ruler, is no wonder because we are combining two distinct concepts. By the way, except for jet stream allowances, this model is the model of airplane travel!

While it is clear that the distance between two points can be determined, we generate no specific method for finding that distance. Unless a circular geoboard is available, we recommend finding the desired distance by using string or wire and then measuring. Although the method is cumbersome, we feel that it provides necessary preparation to better comprehend the spherical distance and concept of a line on a sphere.

2.2.3 EUCLIDEAN DISTANCE

In this instance we return to our usual understanding of a plane as a "flat surface" and have a better concept of what is implied by distance—sometimes referred to as "as the crow flies" (Figure 2.2e). This originates from the fact that here, as in spherical distance considerations, two points determine a line. As with the spherical model, the line between distinct points P and Q can be found by appropriate use of string, wire, or a rubber band.

Figure 2.2e

The calculating of Euclidean distance is a very simple matter, but it depends upon prerequisite knowledge of two-dimensional, coordinate graphing and the *Pythagorean Theorem*, which states:

> "In a right triangle, the sum of the squares of the lengths of the legs equals the square of the length of the hypotenuse."

Figure 2.2f shows the relationship, which is stated $a^2 + b^2 = c^2$ where a, b, and c represent the Euclidean lengths of the indic..ed sides of the right triangle.

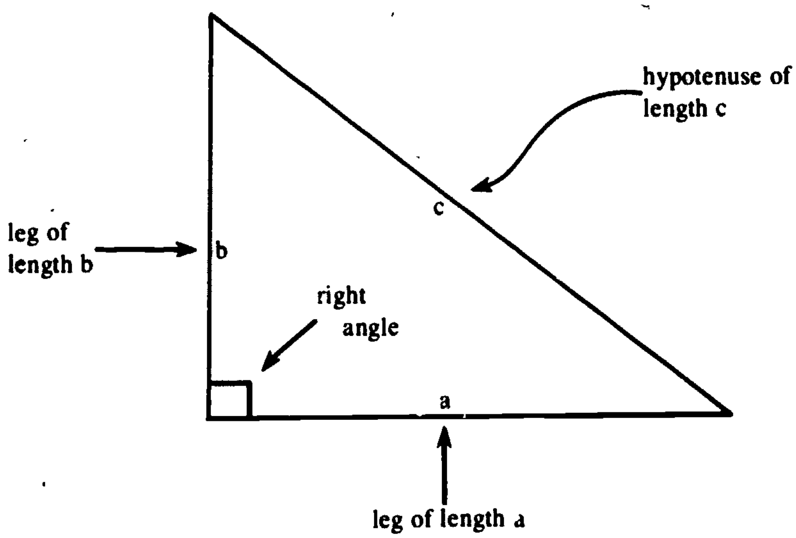
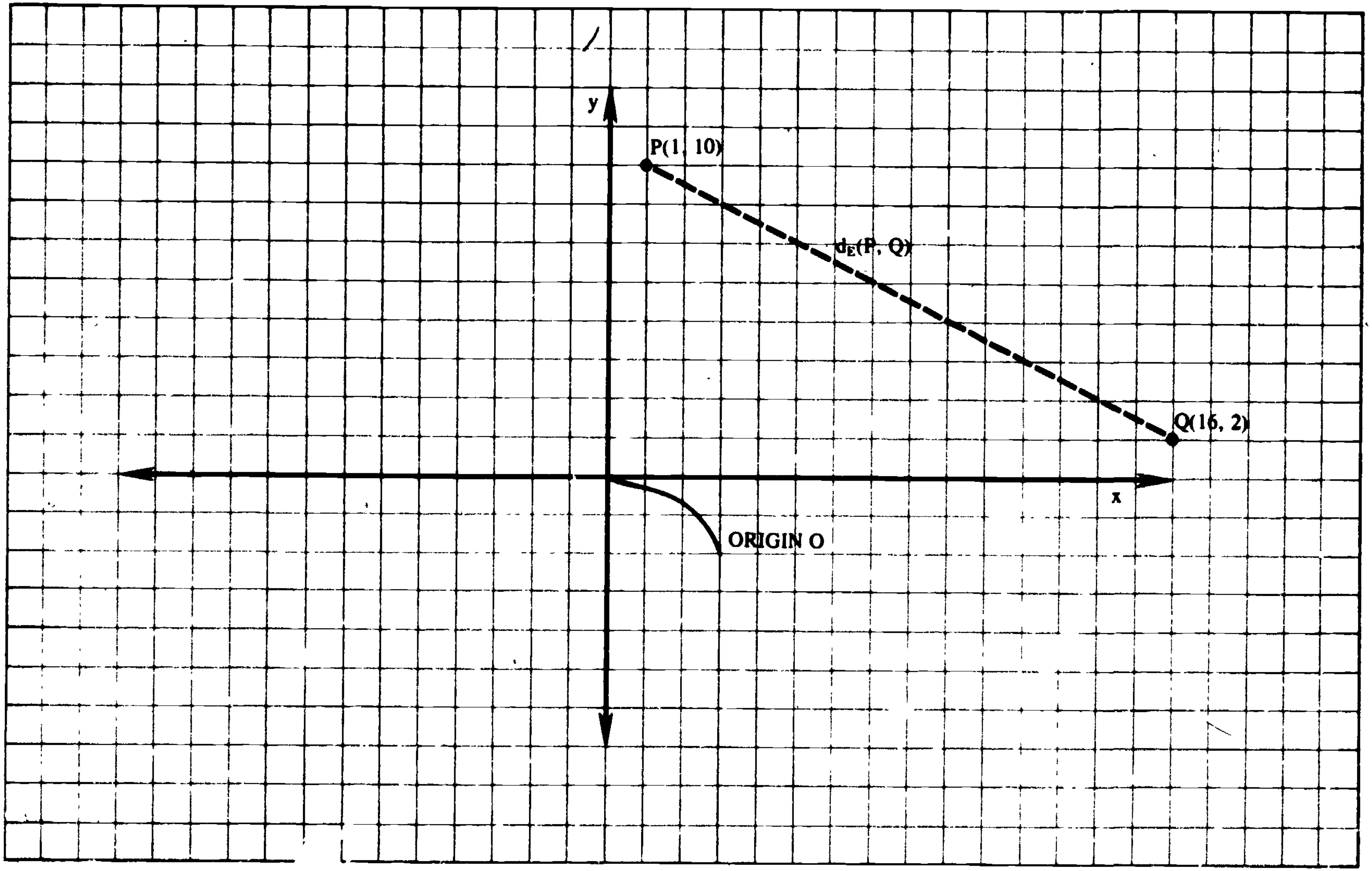


Figure 2.2f

The use of a two-dimensional coordinate system is pictured in Figure 2.2g. We must have an origin 0, and two perpendicular number lines, say x and y. With these specified, we name point P as P(1, 10). The *coordinates* of P are called (1, 10) and indicate P is 1 unit to the right and 10 units up. Similarly point Q is named as Q(16, 2) and the Euclidean distance between P and Q, $d_E(P\ O)$, is indicated by the dotted line.

¼" squares



SOURCE: Science and Mathematics Teaching Center
University of Wyoming

Figure 2.2g

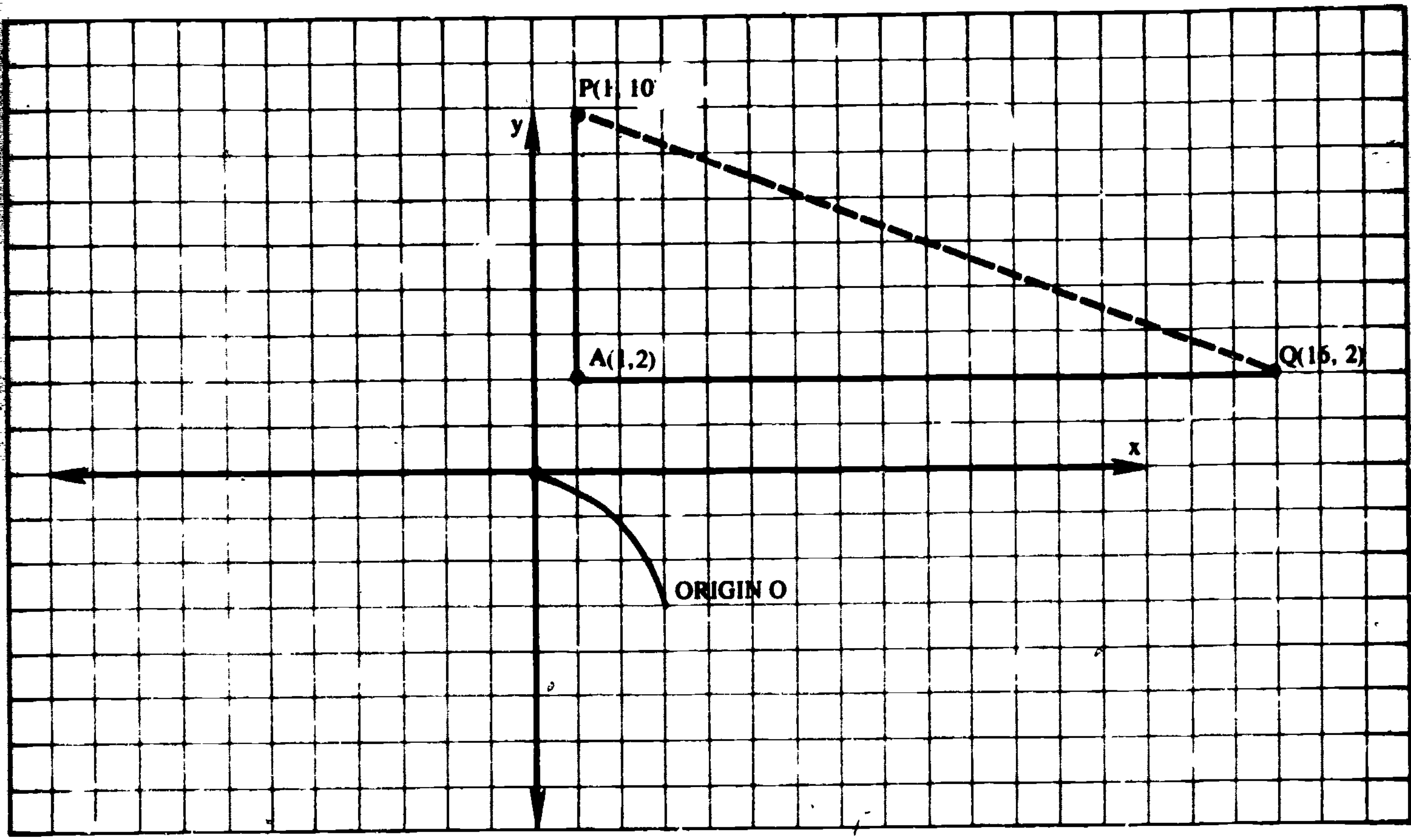


Figure 2.2h.

When P, Q and the dotted line are given, a "natural" right triangle, $\triangle APQ$, emerges (Figure 2.2h) with A(1, 2), P and Q as the vertices. Furthermore, using the coordinates of P, Q, and A, we can determine the Euclidean lengths of the legs of $\triangle APQ$, that is, $d_E(A, P) = 8$ and $d_E(A, Q) = 15$. Using the Pythagorean Theorem we know $(d_E(P, Q))^2 = (d_E(A, P))^2 + (d_E(A, Q))^2$ hence $(d_E(P, Q))^2 = 8^2 + 15^2 = 289$—so, $(d_E(P, Q))^2 = 289$ and $d_E(P, Q) = 17$.

Of course, the distance between two points will not always be a whole number. For instance, if P(1, 3) and Q(4, 5) are given, we find $(d_E (P, Q))^2 = (3)^2 + (2)^2 = 13$—so $(d_EP, Q))^2 = 13$ and $d_E(P, Q) = \sqrt{13}$. The use of a calculator will enable students to calculate this to the degree of accuracy desired. Also, younger children might use string and rulers in conjunction with the graph paper to obtain a fairly accurate measurement between the two points (keeping in mind the units used on the graph paper and ruler).

For a general way to compute Euclidean distance we examine a case where P(x, y) and $Q(x_1, y_1)$ are given (Figure 2.2i). After drawing right triangle APQ, we find $d_E(A, Q) = (x - x_1)$ and $d_E(A, P) = (y - y_1)$. Hence, $(d_E(P, Q))^2 = (x - x_1)^2 + (y - y_1)^2$ and $d_E(P, Q) = \sqrt{(x - x_1)^2 + (y - y_1)^2}$. This is the general form for calculating Euclidean distance.

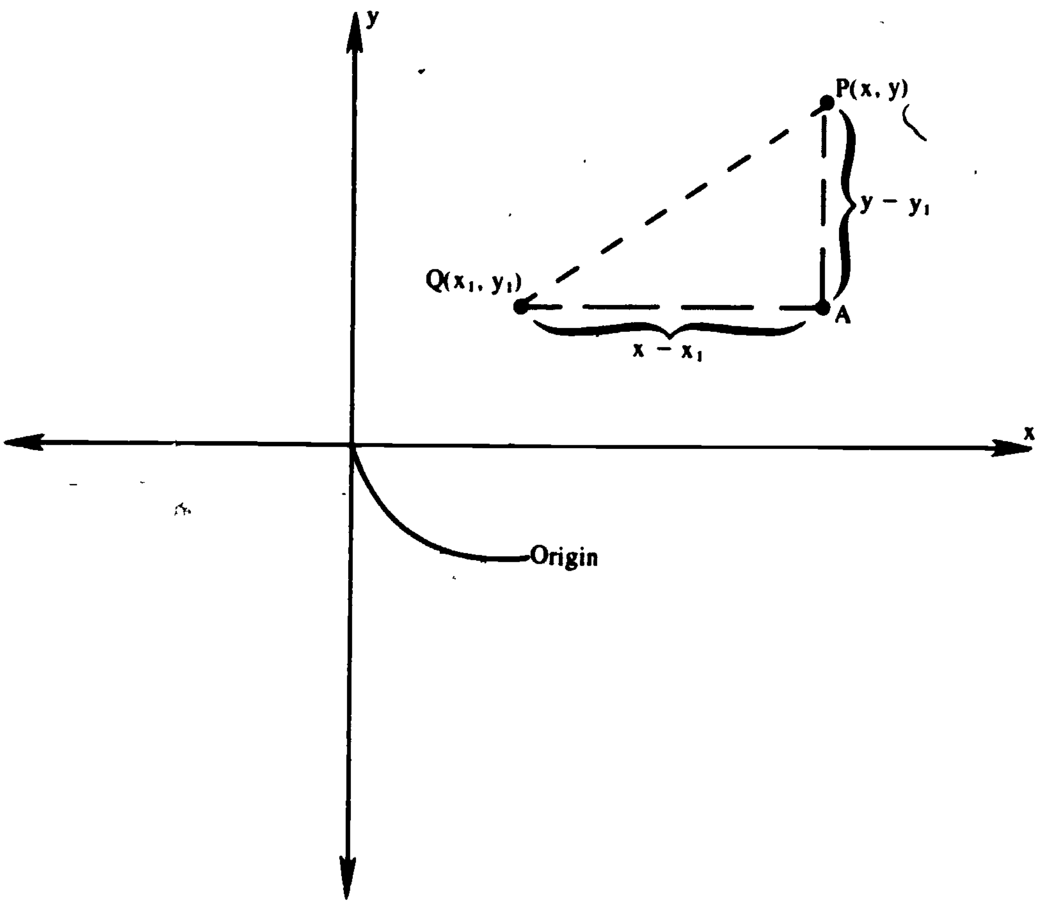


Figure 2.2i

Exercises

Using the above as a model, complete the following exercises.

1. (a) If P(2, 1) and Q(−2, −2) are specified, find $d_E(P, Q)$.

   (b) If P(−4, −1) and Q(1, 11) are specified, find $d_E(P, Q)$.

   (c) If P(5, −1) and Q(−3, 6) are specified, find $d_E(P, Q)$.

2. If P(3, 7) is given, find eight points that are 5 units from P. Place these eight points on a graph and make a conjecture concerning their location.

Note: In exercises 1(a), (b), (c) above, $d_E(P, Q)$ is 5, 13 and $\sqrt{113}$ units, respectively. In exercise 2, the points which are easy to find are (3, 2), (6, 3), (7, 4), (8, 7), (7, 10), (6, 11), (3, 12), (0, 11), (−1, 10), (−2, 7), (−1, 4) and (0, 3). When graphed, these points lie on a circle of radius 5 whose center is (3, 7).

2.2.4 TAXICAB DISTANCE

If Euclidean distance has the connotation of "as the crow flies," then taxicab distance can be classified as "as the taxi travels." Again, we use our connotation of a plane as a "flat" surface, but measure distance differently. Consider Figure 2.2j, where P(1, 3) and Q(−2, −2) are given and we consider the grid as a part of town, each unit representing a block. The distance between P and Q is considered the distance an "honest" taxi would travel horizontally or vertically from one point to the other. The dotted segments suggest two such possible routes and clearly the distance is eight blocks in each case. We denote the taxi distance as d_T, and for this example note $d_T(P, Q) = 8$.

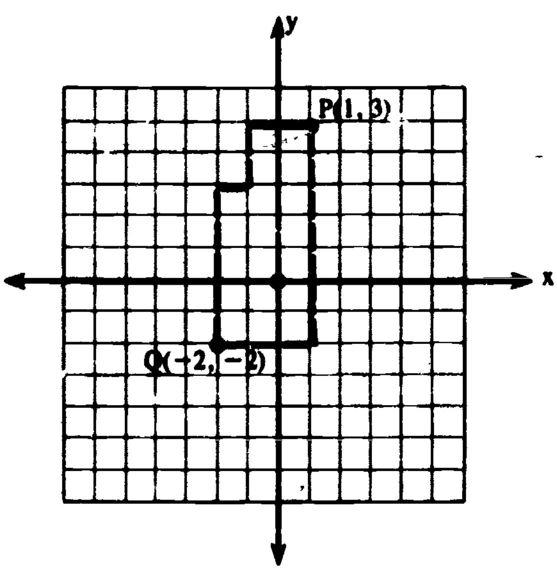


Figure 2.2j

33

The taxicab distance is perhaps the distance that is most familiar to kids at an early age. Although not all cities are laid out on a rectangular grid exclusively, almost every city has an area as such. This prompts us to use phrases as, "It is seven blocks to school," "It is eight blocks to the store," "My friend lives two blocks away," and "It is four blocks to the park." All of these phrases are indicating distance measured by the taxicab method. This method of calculating distance is also appropriate in many rural areas, where the rectangular grid, based upon one mile instead of one block, was legislated by the government. The ambitious teacher will make wise use of the students' understanding of the above uses of the taxicab distance.

We are now in position to give a formal definition for the taxicab distance between two points. The only concept needed is absolute value of a number a, denoted by $|a|$. The absolute value concept is best illustrated by an example. For instance,

$$|-8| = 8, \quad |13| = 13, \quad |-89| = 89 \text{ and } |93| = 93.$$

In general, the absolute value of a number is itself or its additive inverse (opposite), whichever is larger. If P(3, 5) and Q(1, 8) are two points, the taxicab distance between them is found by finding how far apart the first coordinates are, how far apart the second coordinates are, then adding the two results (in this case $|(3-1)| = 2$ and $|(5-8)| = 3$ which, when added, is equal to 5). This allows us to "drive" along the horizontal distance, then "drive" along the vertical distance to find the total distance between two points.

Formally, if $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points, we define the taxicab distance between P and Q as $d_T(P, Q) = |(x_1 - x_2)| + |(y_1 - y_2)|$. We make this definition to insure that there is a definite taxicab distance between any two points in a plane, whether or not they are located at a "street corner." In the exercises which follow, we use only points whose coordinates are integers. Hence, graph paper can and should be utilized to determine the solutions, as all points are located at "street corners."

Exercises

1. (a) If P(1, 5) and Q(3, −2) are given, find $d_T(P, Q)$.

   (b) If P(−3, 6) and Q(4, 13) are given, find $d_T(P, Q)$.

   (c) If P(1, −3) and Q(−8, 7) are given, find $d_T(P, Q)$.

   (d) If P(−3, −4) and Q(2, −3) are given, find $d_T(P, Q)$.

2. If P(3, 8) is given, find as many points as you can that are 5 units away from P. Place these points on a graph and make a conjecture concerning their location. How does this compare with the solution in Exercise 2 of Section 2.2.3?

3. If P(1, 5) and Q(3, 9) are given, we know $d_T(P, Q) = 6$. How many "paths" could a taxi take in traveling between P and Q?

4. Find two points P and Q such that $d_T(P, Q) = d_E(P, Q)$.

5. For each pair of points P and Q calculate both $d_T(P, Q)$ and $d_E(P, Q)$. Conjecture.

   (a) P(1, 5), Q(3, 7), $d_T(P, Q)$ = ______; $d_E(P, Q)$ = ______
   (b) P(3, 8), Q(−1, 11), $d_T(P, Q)$ = ______; $d_E(P, Q)$ = ______

(c) P(−1, 6), Q(4, −6), $d_T(P, Q) =$ ______; $d_E(P, Q) =$ ______
(d) P(4, −3), Q(0, 0), $d_T(P, Q) =$ ______; $d_E(P, Q) =$ ______

6. In 5(b) and 5(d), $d_T(P, Q) = 7$ and $d_E(P, Q) = 5$.
   (a) Can you find P and Q where $d_T(P, Q) = 7$ and $d_E(P, Q) \neq 5$, and
   (b) Can you find P and Q where $d_T(P, Q) \neq 7$ and $d_E(P, Q) = 5$?

7. What is meant by an "honest" taxi?

Note: In exercises 1(a), (b), (c), (d), $d_T(P, Q) = 9$, 14, 19, and 6 respectively.

In exercise 2, the points (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (8, 8), (7, 9), (6, 10), (5, 11), (4, 12), (3, 13), (2, 12), (1, 11), (0, 10), (−1, 9), (−2, 8), (−1, 7), (0, 6), (1, 5) and (2, 4) are all 5 units from (3, 8). This when graphed does not outline a circle as the points did in exercise 2 of section 2.2.3, but instead outlines a "diamond" whose "center" is (3, 8).

In exercise 3, there is a large number of paths. However, using graph paper and considering an "honest" taxi, we see that there are 15 paths. You should try to find all of them.

In exercise 4, we see that any two points on the same horizontal or vertical line are the same taxi and Euclidean distance apart.

In exercise 5, note that the taxi distance between two points is always equal to or larger than the Euclidean distance. When considering "as the crow flies" and "as the taxi travels," this is easily understood.

In exercise 6(a), consider p(0, 0) and Q(0, 7), and in exercise 6(b) consider P(0, 0) and Q(0, 5). These are only one of many solutions for each.

In exercise 7 the discussion of what constitutes an "honest" taxi should precede any work students perform with taxi distance.

Activity

If you do not have access to a circular geoboard, we provide the directions for construction of a circular geoboard having maximum distance 10, that is, M = 10. The use of a compass and protractor is necessary, and this construction can be modified to accommodate other values of M.

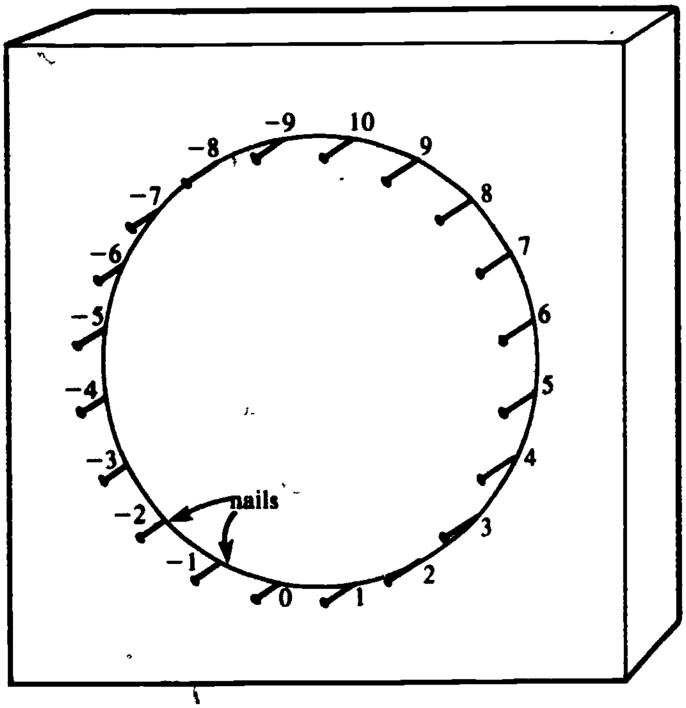
1. Obtain a 25-centimeter-square piece of plywood, one-half inch thick.

2. Obtain 20 finishing nails about 1 inch long.

3. Find the center of the piece of plywood and construct a circle of radius 10 centimeters.

4. Using a protractor (and great care) divide the circle into 20 equal parts and place your nails there. Choose one point as the origin, 0, and assign positive coordinates to the nails on the right and negative coordinates to nails on the left of the origin.

5. Below is what the finished product should look like.



2.3
Congruence

2.3.1 WHAT IS CONGRUENCE?

In a very general way, two geometric figures are congruent if they have the same size and shape. Some teachers explain that two geometric figures are congruent if one of them can be "moved onto" the other one so that they fit together exactly. This approach suggests a technique or way of "moving" or "transforming" these figures in the plane. In our development, we have not yet explored these ideas, but we shall look deeply into them later on.

Suppose we have two segments, say AB and XY, of equal length as given in Figure 2.3a.

Figure 2.3a

Since segments AB and XY consist of different sets of points, we know that *as sets of points* they are not equal; that is, $\overline{AB} \neq \overline{XY}$. So, we find ourselves in a situation where $\overline{AB} \neq \overline{XY}$, but AB = XY. This might seem a bit confusing to you at first, but it is analagous to the situation involving two different persons of the same height. It is incorrect to conclude that these individuals are the same individuals because they are the same height. To the contrary, all that we can say about them is that they are simply the same height! Returning to segments AB and XY given in Figure 2.3a, we cannot correctly say that they are the same or equal—only that they are the same length!

So, we say two segments are CONGRUENT if they have the same length. That is, segments AB and XY are congruent if AB = XY; in this instance, we write $\overline{AB} \cong \overline{XY}$.

Since we have made such an issue of the difference between equality and congruence of line segments, is it ever appropriate to write $\overline{AB} = \overline{XY}$? AB ≅ XY?

In light of this discussion and our previous developments, it is reasonable to attempt to define some analogous relation between angles. Suppose we have two angles, say ABC and XYZ, of equal measure as given in Figure 2.3b.

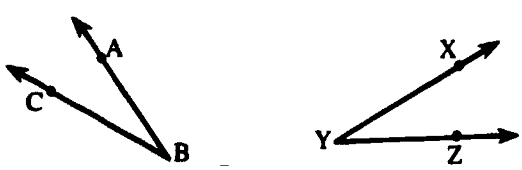


Figure 2.3b

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Since angles ABC and XYZ are different *as sets of points*, they are not equal; that is, $\measuredangle ABC \neq \measuredangle XYZ$. So, we find ourselves in the situation where $\measuredangle ABC \neq \measuredangle XYZ$ but $m\measuredangle ABC = m\measuredangle XYZ$ just as we did with segments earlier.

So, we say two angles are CONGRUENT if they have the same measure. That is, angles ABC and XYZ are congruent if $m\measuredangle ABC = m\measuredangle XYZ$; in this instance, we write $\measuredangle ABC \cong \measuredangle XYZ$.

Is it ever appropriate to write $\measuredangle ABC = \measuredangle XYZ$? $m\measuredangle ABC \cong m\measuredangle XYZ$?

Now we are ready to introduce the congruence relation for triangles. Suppose we have two triangles, say ABC and XYZ, as given in Figure 2.3c.

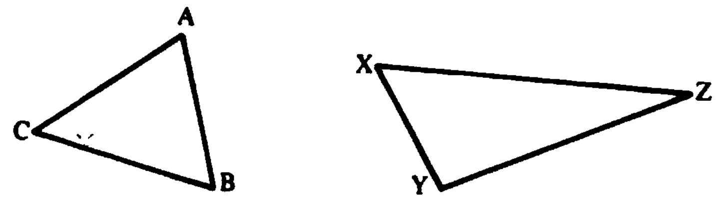

Figure 2.3c

Consider the sets containing the vertices of these triangles; let $S=\{A, B, C\}$ and $T=\{X, Y, Z\}$. Suppose that a one-to-one correspondence exists between sets S and T described by:

A "corresponds to" X, B "corresponds to" Y, C "corresponds to" Z

In abbreviated form, we write these correspondences as

$$A \longleftrightarrow X,\ B \longleftrightarrow Y,\ C \longleftrightarrow Z$$

Or, we agree to shorten this abbreviation still further to read

$$ABC \longleftrightarrow XYZ$$

So, whenever we write $ABC \longleftrightarrow XYZ$, we mean that a one-to-one correspondence exists between the vertices of the triangles ABC and XYZ defined to mean $A \longleftrightarrow X$, $B \longleftrightarrow Y$, $C \longleftrightarrow Z$. What does $ACB \longleftrightarrow XZY$ describe? $CAB \longleftrightarrow ZYX$? The implication here is that the *order* in which the matching pairs appear is irrelevant in describing the correspondence.

Another observation is that the one-to-one correspondence between the vertices of the triangles imposes a one-to-one correspondence between the sides of the triangles, and it imposes another one-to-one correspondence between the angles of the triangles. For triangles ABC and XYZ in Figure 2.3c, under the one-to-one correspondence $ABC \longleftrightarrow XYZ$, we have one-to-one correspondences for the sides and angles given by

$$\overline{AB} \longleftrightarrow \overline{XY},\ \overline{BC} \longleftrightarrow \overline{YZ},\ \overline{AC} \longleftrightarrow \overline{XZ} \text{ and}$$
$$\measuredangle A \longleftrightarrow \measuredangle X,\ \measuredangle B \longleftrightarrow \measuredangle Y,\ \measuredangle C \longleftrightarrow \measuredangle Z$$

Pairs of corresponding sides ($\overline{AB}$ and $\overline{XY}$, for example) and corresponding angles ($\measuredangle A$ and $\measuredangle X$, for example) are commonly referred to as CORRESPONDING PARTS of the triangles.

We are now ready to define the congruence relation for triangles. Consider two triangles, say ABC and XYZ, and a one-to-one correspondence between their vertices, say $ABC \longleftrightarrow XYZ$. If *every* pair of corresponding sides is congruent and every *pair* of corresponding angles is congruent, then the two triangles are CONGRUENT and the

correspondence ABC $\longleftrightarrow$ XYZ, is called a CONGRUENCE. In this instance, we write $\triangle ABC \cong \triangle XYZ$.

When we write $\triangle ABC \cong \triangle XYZ$, we mean that a one-to-one correspondence exists between the vertices of triangles ABC and XYZ defined by ABC $\longleftrightarrow$ XYZ. Furthermore, it means that each of the corresponding sides of these triangles is congruent; that is, $\overline{AB} \cong \overline{XY}$, $\overline{BC} \cong \overline{YZ}$, $\overline{AC} \cong \overline{XZ}$. And, each of the corresponding angles of these triangles is congruent; that is, $\measuredangle A \cong \measuredangle X$, $\measuredangle B \cong \measuredangle Y$, $\measuredangle C \cong \measuredangle Z$.

We can now ask some familiar questions. Is it ever appropriate to write $\triangle ABC = \triangle XYZ$? $\triangle XYZ \cong \triangle ABC$?

We hope it is clear from our definition of triangle congruence that the following concepts are involved:

(i) a one-to-one correspondence between the vertices (also the sides and angles) of the triangles,

(ii) congruence of the corresponding sides of the triangles, and,

(iii) congruences of the corresponding angles of the triangles.

The crucial prerequisite concepts involved are one-to-one correspondence, segment congruence (length or, more generally, distance), and angle congruence (measure). Teachers should be particularly alert to activities that encourage the achievement of these prerequisites so that a synthesis of them, i.e., triangle congruence, is the desired outcome. The most frequently omitted condition is one-to-one correspondence; to help you in this area we have included exercises 4–6 at the end of the section.

Now, a very provocative question. Is there some minimal combination of side and/or angle congruence conditions that will guarantee triangle congruence? You may recall from your earlier studies that if certain combinations of only three segment and/or angle congruences can be shown to exist, then the remaining three congruences needed to guarantee the triangle congruence (from the definition) must also hold. The Activity at the end of this section is directed at this issue.

Exercises

1. Show that every segment is congruent to itself. This says that segment congruence satisfies the REFLEXIVE PROPERTY.

   If $\overline{AB} \cong \overline{XY}$, then show that $\overline{XY} \cong \overline{AB}$. This result says that segment congruence satisfies the SYMMETRIC PROPERTY.

   If $\overline{AB} \cong \overline{XY}$ and $\overline{XY} \cong \overline{PQ}$, then show that $\overline{AB} \cong \overline{PQ}$. This says that segment congruence satisfies the TRANSITIVE PROPERTY.

   If a particular relation satisfies the reflexive property, symmetric property, and transitive property, we say the relation is an EQUIVALENCE RELATION. Since segment congruence satisfies these three properties, we say segment congruence is an equivalence relation.

2. Show that angle congruence is an equivalence relation.

3. Show that triangle congruence is an equivalence relation.

4. Draw any triangle and label the vertices A, B, C.

   (i) How many one-to-one correspondences between triangle ABC and itself are there? Write each of them down and see if you can determine a scheme for representing them.

   (ii) Which of the correspondences given in part (i) are congruences when AB = BC?

   (iii) Which of the correspondences given in part (i) are congruences when AB = BC = AC?

5. Consider the four-sided figure given in Figure 2.3d below. How many one-to-one correspondences are there between this figure and itself? Write each of them down and see if you can determine a scheme for representing them.

6. Consider the five-pointed star given in Figure 2.3e below. How many one-to-one correspondences are there between the star and itself? As in exercises 4 and 5, write each of them down and see if you can determine a scheme for representing them.

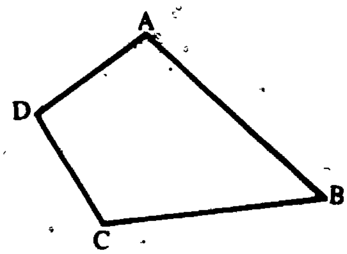

Figure 2.3d

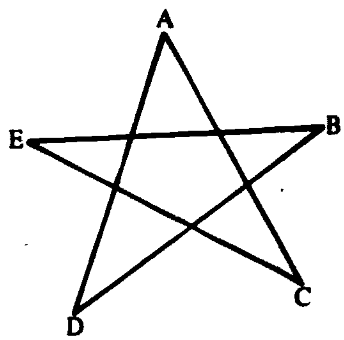

Figure 2.3e

Activity

The intent of this activity is to help you understand that if certain combinations of only three segment congruences and/or angle congruences can be shown to exist, then the three remaining congruences needed to guarantee the triangle congruence (from the definition) must also hold. You may use a ruler, protractor, or compass to complete the activity, remembering that some of the constructions you will be asked to make are impossible with the information given. Assume we are completing this activity in the Euclidean model.

(i) Can you construct a triangle ABC having AB = 2.5 cm, AC = 1.5 cm, and $m\angle A = 35$? Would all triangles constructed from these directions be congruent to yours?

(ii) Can you construct a triangle XYZ with XY = 2 cm, $m\angle X = 45$, and $m\angle Y = 60$? Would all triangles constructed from these directions be congruent to yours?

(iii) Can you construct a triangle ABC having AB = 3 cm, AC = 2 cm, and BC = 3.5 cm? Would all triangles constructed from these directions be congruent to yours?

(iv) Can you construct a triangle ABC with $m\angle A = 40$, $m\angle B = 80$, and $m\angle C = 60$? Would all triangles constructed from these directions be congruent to yours?

(v) You cannot construct a triangle ABC having AC = 4 cm, AB = 2 cm, and BC = 1 cm. Why? Can you give other lengths for AB, AC, and BC so that no triangle can be formed?

(vi) *Using only your ruler*, construct any triangle that has no two congruent sides.

Describe the procedure you would use to construct a second triangle that is congruent to the first.

Is there more than one way to obtain this second triangle?

How many of the six parts of the first triangle were used in forming the second triangle?

What is the *least* number of congruent parts necessary to insure that the two triangles are congruent?

(vii) Construct triangle XYZ with $m\measuredangle X = 40$, XZ = 3 cm, and YZ = 2 cm. Now construct triangle ABC with $m\measuredangle A = 40$, AC = 3 cm, and BC = 2 cm. These two triangles need not be congruent. Why?

(viii) Is it possible to assign measures to angles and lengths to sides so that no triangle is determined? Explain.

2.3.2 TRIANGLE CONGRUENCE CRITERIA

We saw in the last section that if certain combinations of only three segment and/or angle congruences of two triangles can be shown to exist, then the triangles are congruent. One of the most common of these side/angle combinations, usually referred to as "Side-Angle-Side" and denoted "SAS", is:

> If between two triangles there is a one-to-one correspondence between the vertices in which two sides *and* the included angle of one triangle are congruent, respectively to the corresponding two sides *and* the included angle of the other triangle, then the two triangles are congruent.

This idea is shown in Figure 2.3f, where we have triangles ABC and PQR and the one-to-one correspondence ABC $\longleftrightarrow$ PQR with $\overline{AB} \cong \overline{PQ}$, $\measuredangle B \cong \measuredangle Q$, and $\overline{BC} \cong \overline{QR}$. That is, we have two triangles with a one-to-one correspondence defined between their vertices and two sides and the included angle of one triangle congruent, respectively, to the two corresponding sides and included angle of the other triangle. So, all of the requirements seem to be satisfied and we are justified in concluding that $\triangle ABC \cong \triangle PQR$.

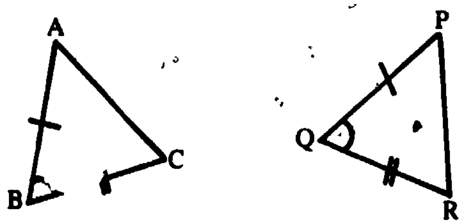

Figure 2.3f

We could verify that these triangles are indeed congruent by showing that $\overline{AC} \cong \overline{PR}$, $\measuredangle A \cong \measuredangle P$, and $\measuredangle C \cong \measuredangle R$ by means of a ruler and protractor. However, if we accept the SAS statement, this last activity is not necessary.

The "SAS" congruence criteria do afford us much economy in establishing triangle congruences. As a matter of fact, they cut our work in half!

Now, let's enumerate all of the possible analagous combinations of sides/angles under a one-to-one correspondence between two given triangles. They are:

Angle-Side-Angle (ASA)
Side-Side-Side (SSS)
Angle-Angle-Side (AAS)
Side-Side-Angle (SSA)
Angle-Angle-Angle (AAA)

Which of these, if any, guarantees that the triangles so related are congruent? The answer to this question is the focus of the remainder of this section. You may wish to return to the Activity in Section 2.3.1. for some welcome hints.

First, let's consider a precise statement of "Angle-Side-Angle."

> If between two triangles there is a one-to-one correspondence between the vertices in which two angles *and* the included side of one triangle are congruent, respectively, to the corresponding two angles *and* the included side of the other triangle, then the two triangles are congruent.

In Figure 2.3g, there is a one-to-one correspondence between the vertices of triangles RST and XYZ defined by PST $\longleftrightarrow$ XYZ and $\measuredangle R \cong \measuredangle X$, $\overline{RS} \cong \overline{XY}$, and $\measuredangle S \cong \measuredangle Y$.

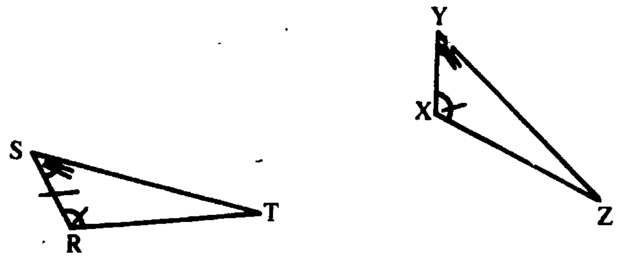

Figure 2.3g

We could verify that $\overline{RT} \cong \overline{XZ}$, $\measuredangle T \cong \measuredangle Z$, and $\overline{ST} \cong \overline{YZ}$ using ruler and protractor methods and thereby conclude that $\triangle RST \cong \triangle XYZ$. But, such an approach would allow us to conclude that only these particular triangles (triangles RST and XYZ) are congruent, not that *any* two triangles satisfying the ASA relationship are congruent. We want to be assured that *any* two triangles satisfying the stated ASA relationship must be congruent.

We are able to establish this as a consequence of the SAS criteria assumed earlier. The actual proof of the ASA criteria is not presented here but several hints are given in Exercise 1 for those who wish to try.

If we were to formulate a statement for "Side-Side-Side," we would have:

> If between two triangles there is a one-to-one correspondence between the vertices in which the three sides of one triangle are congruent, respectively, to the three sides of the other triangle, then the triangles are congruent.

Let's assume this statement with respect to triangles ABC and DEF in Figure 2.3h under the correspondence ABC $\longleftrightarrow$ DEF where $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\overline{AC} \cong \overline{DF}$.

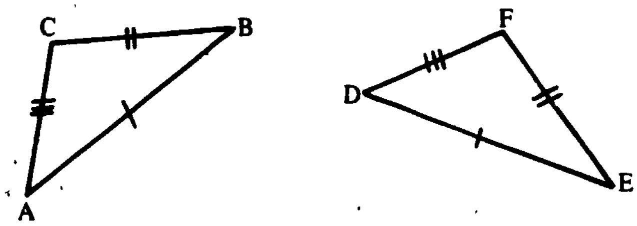

Figure 2.3h

We could verify that these two triangles are congruent by using a protractor to measure any two of the corresponding angles and appealing to the ASA criteria for congruence. And, this measurement approach is appropriate with children who are experiencing these criteria for the first time.

The actual proof is not included here, but it is similar to the argument used for establishing the ASA criteria.

SAS, ASA, and SSS are the most well known and remembered criteria for triangle congruence. Children are usually introduced to them early and are able to recall them many years later. Before moving on to some of the more specific and more easily forgotten criteria, we should pause to recognize that we did not speak to the proof of the SAS criteria as we did with ASA and SSS. As a matter of fact, we indicated that both ASA and SSS were *consequences* of SAS. But where did SAS come from? How were we able to adopt it as a congruence criterion? It turns out that the SAS criterion must be assumed as it cannot be established as a consequence of earlier statements. We will explore this matter later in the section.

Now, let's formulate a statement for "Angle-Angle-Side."

> If between two triangles there is a one-to-one correspondence between the vertices in which two angles and a side *not included* by those two angles of one triangle are congruent, respectively, to two angles and a side *not included* by those angles of the second triangle, then the triangles are congruent.

In Figure 2.3i, consider the correspondence ABC $\longleftrightarrow$ PQR between triangles ABC and PQR with $\angle A \cong \angle P$, $\angle C \cong \angle R$, and $\overline{BC} \cong \overline{QR}$.

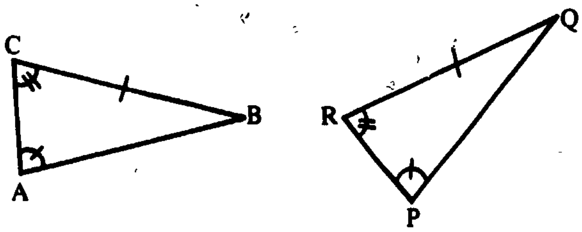

Figure 2.3i

If the AAS statement provides criteria for congruence, then $\triangle ABC \cong \triangle PQR$. You should examine several similar situations involving triangles with two angles and a side *not included* between the angles in an effort to determine if we can include AAS in our list

of triangle congruence criteria. What do you thin'.?

Now, let's consider the triangles given in Figure 2.3j and the correspondence $PQR \leftrightarrow XYZ$ with $\overline{PQ} \cong \overline{XY}$, $\overline{PR} \cong \overline{XZ}$, and $\angle R \cong \angle Z$.

Figure 2.3j

We have a one-to-one correspondence between the vertices of triangles PQR and XYZ in which two sides and an angle *not included* by these sides of one triangle are congruent, respectively, to two sides and an angle *not included* by those sides of the second triangle. This is the basis for the statement we would like to include as the SSA criteria. Are triangles PQR and XYZ congruent? NO! And, we can arrive at this conclusion for many reasons, one being that the corresponding angles P and X are not congruent. This is a sufficient reason, but what are some others?

This means that our candidate for SSA criteria is an unacceptable triangle congruence criterion. If we study the triangles in Figure 2.3j a little more closely, we may be able to find some clues as to why our SSA candidate failed. In particular, study the measures of angles P and X. If these measures were the same, then the triangles would be congruent (by ASA) and eac. would have exactly one right angle at Q and Y, respectively. Why?

What we have discovered is that our SSA candidate holds only for certain types of triangles. Yes, only for right triangles. Now, we can present the SSA congruence criteria:

> If between two right triangles there is a one-to-one correspondence between the vertices in which the right angle, side opposite the right angle, and another side of one triangle are congruent, respectively, to the right angle, side opposite the right angle, and a side of the second triangle, then the triangles are congruent.

The SSA criteria for right triangles are illustrated in Figure 2.3k where $PQR \longleftrightarrow XYZ$ is the correspondence between the vertices of triangles PQR and XYZ and $\angle Q \cong \angle Y$, $\overline{PR} \cong \overline{XZ}$, and $\overline{QR} \cong \overline{YZ}$. By the SSA criteria for right triangles, $\triangle PQR \cong \triangle XYZ$.

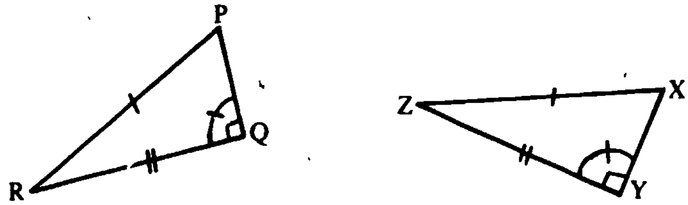

Figure 2.3k

The only remaining combination of three side/angle congruences is "Angle-Angle-Angle" or "AAA". It should be clear that if three angles of one triangle are congruent, respectively, to three angles of another triangle, the triangles need not be congruent. This is shown in Figure 2.3l where $ABC \longleftrightarrow DEF$ is the correspondence between triangles ABC and DEF and $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$. But, none of the sides of these triangles is congruent, so the triangles cannot be congruent.

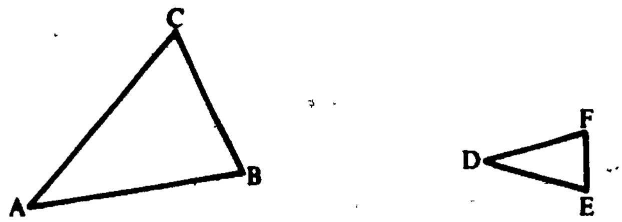

Figure 2.3l

The following table summarizes our findings for all possible combinations of three sides/angle congruences and whether or not they guarantee congruence criteria for the triangles.

Combinations of Sides/Angles	Triangle Congruence Criteria	Type of Triangle	Basis for Including in Our System
SAS	Yes	all	assumed
ASA	Yes	all	proved
SSS	Yes	all	proved
AAS	Yes	all	proved
SSA	Yes	right triangles	proved
AAA	No	—	—

Now let's examine each of these criteria in terms of the models presented earlier.

Euclidean Model

Recall, we interpret a point as an ordered pair of real numbers and define the distance between two points A (x_1, y_1) and B (x_2, y_2) as $d_E(A, B)$ according to the formula,

$$d_E(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This distance is really the length of the side of the triangle ABC opposite the right angle C in Figure 2.3m.

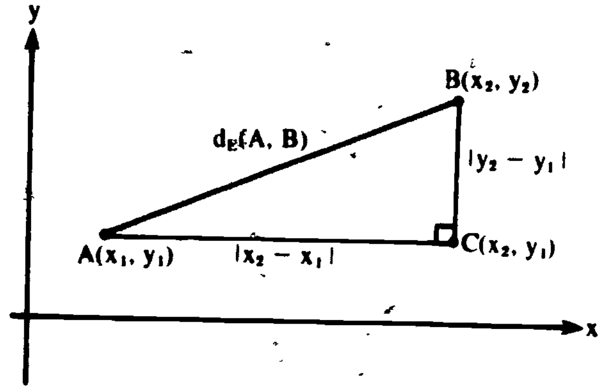

Figure 2.3m

If we now interpret our major definitions with respect to this distance, we say

segments PQ and XY are CONGRUENT if $d_E(P, Q) = d_E(X, Y)$.

angles P and X are CONGRUENT if $m\measuredangle P = m\measuredangle X$ (no change since distance is the only concept affected).

triangles PQR and XYZ are CONGRUENT if there exists a 1-1 correspondence between the vertices (also sides and angles) of the triangles, each of the corresponding angles is congruent, and each of the corresponding sides is congruent. That is, $PQR \longleftrightarrow XYZ$; $\measuredangle P \cong \measuredangle X$, $\measuredangle Q \cong \measuredangle Y$, $\measuredangle R \cong \measuredangle Z$; $d_E(P, Q) = d_E(X, Y)$, $d_E(P, R) = d_E(X, Z)$, $d_E(Q, R) = d_E(Y, Z)$.

Though we never mentioned it, the results in the table really apply to the Euclidean model. Since much of your geometric thinking and intuition is really Euclidean in nature and since we felt it would be easier for you to discover (or rediscover!) these criteria with that frame of reference, we elected that approach. Your intuition may prove to be less valuable, however, as we examine the taxicab and spherical models.

Taxicab Model

In Section 2.2.4 we interpreted a point as an ordered pair of real numbers and defined the distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ as $d_T(A, B)$ according to

$$d_T(A, B) = |x_2 - x_1| + |y_2 - y_1|$$

Recall, this distance was really the sum of lengths (Euclidean distances) of the sides opposite the non-right angles in triangle ABC (Figure 2.3m) or the sum of the vertical and horizontal Euclidean distances between A and B.

We can now interpret the major definitions as we did earlier.

segments LM and RS are CONGRUENT if $d_T(L, M) = d_T(R, S)$.

angles L and R are CONGRUENT if $M\measuredangle L = m\measuredangle R$ (no change since distance is the only concept affected).

triangles LMN and RST are CONGRUENT if there exists a one-to-one correspondence between the vertices (also sides and angles) of the triangles, each of the corresponding angles is congruent, and each of the corresponding sides is congruent. That is, $LMN \longleftrightarrow RST$; $\measuredangle L \cong \measuredangle R, \measuredangle M \cong \measuredangle S, \measuredangle N \cong \measuredangle T$; $d_T(L, M) = d_T(R, S)$; $d_T(L, N) = d_T(R, T)$ and $d_T(M, N) = d_T(S, T)$.

Now we are ready to explore the various abbreviated congruence criteria in terms of the Taxicab model. Let's first look at the SAS statement—Are two triangles related according to the SAS criteria necessarily congruent in this model? The answer is NO! And, the explanation which follows shows why.

Consider triangles ABC and XYZ in Figure 2.3n where A(2, 7), B(2, 1), C(8, 1), X(7, 6), Y(10, 9), and Z(13, 6).

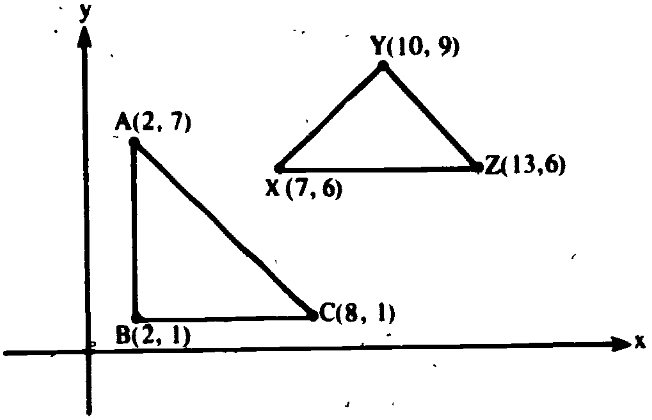

Figure 2.3n

Now, $d_T(A, B) = |2 - 2| + |1 - 7| = 6$ and $d_T(X, Y) = |10 - 7| + |9 - 6| = 6$. So, segments AB and XY are congruent. Since the $m\angle B = 90 = m\angle Y$, angles B and Y are congruent. Furthermore, $d_T(B, C) = |8 - 2| + |1 - 1| = 6$ and $d_T(Y, Z) = |13 - 10| + |6 - 9| = 6$. So, segments BC and YZ are congruent. Thus, we have a one-to-one correspondence between the vertices of triangles ABC and XYZ defined by $ABC \longleftrightarrow XYZ$ and two sides and an included angle of one triangle are congruent, respectively, to two sides and an included angle of the second triangle. Are triangles ABC and XYZ congruent?

No, they are not congruent, since the definition of triangle congruence cannot be satisfied. That is, the remaining corresponding sides (AC and XZ) are not congruent since $d_T(A, C) = 12$ and $d_T(X, Z) = 6$!

This example illustrates an extremely important idea and answers the question of why the SAS criteria were *assumed* earlier. You see, the Euclidean and Taxicab geometries are both models of our postulational system to this point. One could interpret them simultaneously without fear of including results which do not pertain to both. But since we desire to economize and shorten the criteria for triangle congruence in our system, we must look at the impact of this decision upon both models.

The impact of this decision for the Euclidean model is the collection of criteria outlined in the table. Namely, we are able to establish additional congruence criteria from this one "SAS" assumption. From the example above, to make the "SAS" assumption is to eliminate Taxicab geometry as a tenable model of our system. So, if we include the SAS criteria in our list of assumed statements, or postulates, then Euclidean geometry remains as a model and the Taxicab geometry is eliminated.

This also serves to illustrate the role of models in developing mathematical systems. The fewer the assumptions, i.e., assumed statements, the more models which exist. As new assumptions are made and the list of postulates thereby expanded, the number of models is decreased as fewer interpretations of the systems are possible.

Many models are more interesting than others, but they all serve a purpose. The Taxicab model, when presented as an alternative with the ordinary Euclidean distance, is very interesting and it serves as the motivation for the inclusion of the SAS criteria into our system.

Now, we will focus on another model.

Spherical Model

In Section 2.2.2 we interpreted the plane as the surface of a sphere and the distance between two points P and Q (Figure 2.3o), $d_s(P, Q)$ on the sphere is the minimal length along great circle paths having those two points as end points.

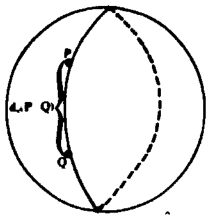

Figure 2.3o

Since the spherical model is so intuitive, one can readily interpret concepts using such concrete objects as a basketball or beach ball; one must also be warned, however, that this very asset can be also a liability at the same time.

Interpreting the major concepts with this definition of distance we say:

segments AB and XY are CONGRUENT if $d_s(A, B) = d_s(X, Y)$.

angles B and Y are CONGRUENT if $m\measuredangle B = m\measuredangle Y$ (no change since distance is the only concept affected).

triangles ABC and XYZ are CONGRUENT if there exists a one-to-one correspondence between the vertices (also sides and angles) of the triangles, each of the corresponding angles is congruent, and each of the corresponding sides is congruent. That is, $ABC \longleftrightarrow XYZ$; $\measuredangle A \cong \measuredangle X$; $\measuredangle B \cong \measuredangle Y$; $\measuredangle C \cong \measuredangle Z$; $d_s(A, B) = d_s(X, Y)$; $d_s(A, C) = d_s(X, Z)$; $d_s(B, C) = d_s(Y, Z)$.

Assuming that SAS is a congruence criterion, we could establish that both ASA and SSS are also congruence criteria (Exercise 1) in this model.

Unfortunately, the AAS statement presented earlier is not acceptable as the following example illustrates. Consider triangle NEF and NFG (Figure 2.3p) where N is the point that is at a distance one-fourth the circumference of a great circle from the line containing E, F, G. Also, consider the correspondence $NEF \longleftrightarrow NFG$. We have $\measuredangle E \cong \measuredangle G$, $\measuredangle F \cong \measuredangle F$, and $\overline{NE} \cong \overline{NG}$ (since $d_s(N, E) = d_s(N, G)$) and hence all of the criteria of the earlier AAS statement satisfied. But, $\triangle NEF \not\cong \triangle NFG$ since clearly $d_s(E, F) \neq d_s(F, G)$. This means that the AAS statement presented earlier does *NOT* guarantee triangle congruence in the spherical model.

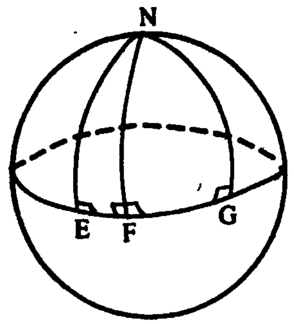

Figure 2.3p

The educational importance of examining alternative models like the sphere becomes more obvious now as we realize we must reexamine the stated AAS criteria and, hopefully, be able to formulate a statement that will be true in this model.

Intuitively, we encountered a problem with the AAS statement because of the special relationship between N (north pole) and the line EG (equator). After much scrutinizing, we would formulate a statement which disallows this relationship. In the spherical model, the AAS criteria for triangle congruence are:

> If between two angles *with sides of length less than one-fourth the circumference of a great circle* there is a one-to-one correspondence between the vertices in which two angles and a side not included by those angles of one triangle are congruent, respectively, to two angles and a side not included by those angles of the second triangle, then the triangles are congruent.

The very process of examining and reexamining one's conjectures in an effort to determine such a statement is the very heart of mathematics. Activities of this kind are extremely difficult for children and adolescents because they involve higher order cognition. The spherical model permits pupils to interact with mathematical ideas in a more concrete setting, and teachers should be prepared to reward pupils exhibiting sound mathematical processes.

Now, let's examine the SSA statement for right triangles mentioned earlier. Is it a congruence criterion in the spherical model? If not, can you provide an example which illustrates this conclusion (Exercise 3)?

Finally, we saw earlier that the AAA relationship stated did not result in congruence criteria (in the Euclidean model). Reconsider the AAA statement with the *spherical model*:

> If between two triangles there is a one-to-one correspondence between the vertices in which three angles of one triangle are congruent, respectively, to the three angles of the other triangle, then the triangles are congruent.

Does the AAA relationship guarantee that the two triangles are congruent (Exercise 4)?

After studying this section, you will, we hope, better understand the criteria for triangle congruence and how these criteria are interpreted in the Euclidean and spherical models. We further hope you understand why the SAS criteria must be postulated and the role that an alternative definition of distance (taxicab distance) played in arriving at that conclusion. Finally, we hope you will use the spherical model to explore mathematical ideas because it is a readily available model and appropriate as we experience technological advances.

Exercises

1. Use the following hints to establish the ASA criteria mentioned in this section (Euclidean and Spherical models) in Figure 2.3q. That is, consider triangle ABC and XYZ and the corresponding ABC $\longleftrightarrow$ XYZ with $\measuredangle A \cong \measuredangle X$, $\overline{AB} \cong \overline{XY}$, and $\measuredangle B \cong \measuredangle Y$. To establish the ASA criteria we must show that $\triangle ABC \cong \triangle XYZ$ in both the Euclidean and Spherical models.

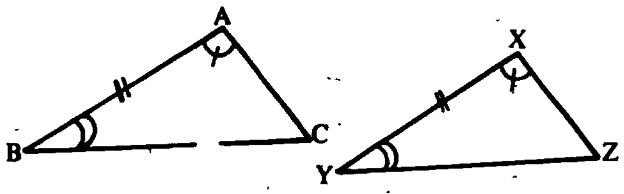

Figure 2.3q

(i) The desired triangle holds if $\overline{BC} \cong \overline{YZ}$. Why?

(ii) Now suppose BC and YZ are not congruent, i.e., they are not the same length. Then one of them is longer than the other, say BC > YZ.

(iii) So, there must be some point D between B and C with BD = YZ (Figure 2.3r).

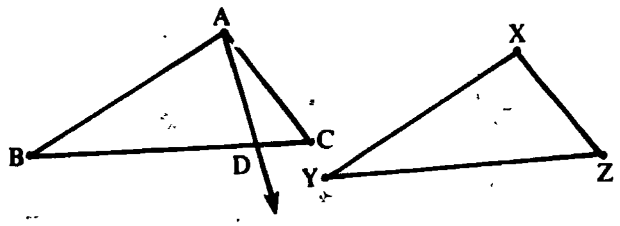

Figure 2.3r

(iv) The correspondence ABD $\longleftrightarrow$ XYZ is a congru... . Why?

(v) $\measuredangle BAD \cong \measuredangle X$ and $\measuredangle BAD \cong \measuredangle BAC$. Why?

(vi) But $m\measuredangle BAD < m\measuredangle BAC$. Why? So, $\measuredangle BAD$ and $\measuredangle BAC$ are not congruent.

(vii) This contradiction implies that segments BC and YZ must be congruent. So, triangles ABC and XYZ are congruent. Why?

2. Provide an argument which establishes SSS as a triangle congruence criterion in both the Euclidean and Spherical models.

3. Give an example which shows that the SSA statement for congruence of right triangles does not hold in the Spherical model.

4. Formulate a statement for congruence of two right triangles in the Spherical model which involves two sides and an angle not included between those sides. (**Note:** The resulting statement would be the SSA congruence criteria in the Spherical model).

5. Show that the AAA statement is a congruence criterion in the Spherical model.

6. Prepare a table that summarizes our findings for all possible combinations of three side/angle congruences and whether or not they gua ntee a congruence criterion for the triangles involved in the Spherical model.

Activities

1. Euclid proposed in Book I (Proposition I) of his monumental work entitled *Euclid's Elements** that a very special relationship existed between the measure of an exterior angle of a triangle and the measures of the remote interior angles associated with that exterior angle. Specifically stated,

   > In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, . . . (p. 316)

   We can see that this statement is erroneous in the Spherical model by considering triangle ABC in Figure 2.3s where m∡BAC < 90, $\overline{AB} \perp \overleftrightarrow{BC}$, $\overline{AC} \perp \overleftrightarrow{BC}$ and the distance between A and the line containing B, C, D is one-fourth the distance of a great circle.

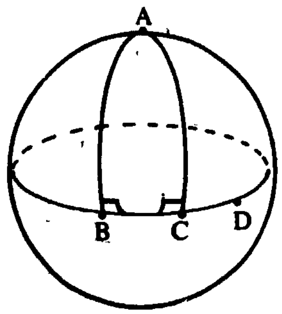

Figure 2.3s

Now, angle ACD is an exterior angle of triangle ABC, and angles BAC and ABC are the two opposite and interior angles (remote interior angles). The relationship mentioned by Euclid clearly does not hold in this case. That is, m∡ACD ≠ m∡BAC + m∡ABC.

What alterations or restrictions could be made in Euclid's statement to produce a statement that explains the relationship between an exterior angle of a triangle and the remote interior angles?

*T. L. Heath (Translation and Commentary), *The Thirteen Books of Euclid's Elements (Volume I)*. New York: Dover Publications, Inc., 1956.

The remaining activities are a bit more advanced and should be interpreted in the Euclidean model.

2. Consider the statement: All triangles are isosceles.

   Criticize the following "proof" of this statement.

   With regard to triangle ABC in Figure 2.3t suppose that ray AO bisects angle BAC, line OM is the perpendicular bisector of segment BC, and the segments OD and OE are perpendicular respectively to segments AB and AC.

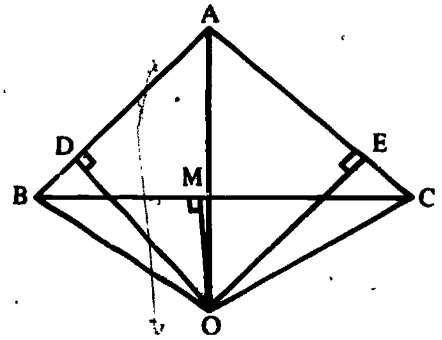

Figure 2.3t

   The correspondence DAO $\longleftrightarrow$ EAO is a congruence (Why?). From this congruence, it follows that $\overline{AD} \cong \overline{AE}$.

   The correspondence BOD $\longleftrightarrow$ COE is a congruence (Why?). Thus, $\overline{DB} \cong \overline{EC}$. It follows that AD + DB = AE + EC, or, AB = AC.

   Hence, triangle ABC is isosceles.

3. Consider the statement: All obtuse angles are right angles.

   Criticize the following "proof" of this statement.

   Suppose angle ABC in Figure 2.3u is obtuse.

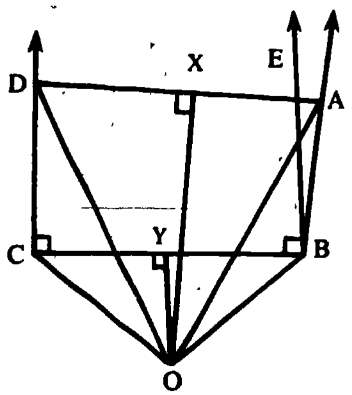

Figure 2.3u

Construct ray BE perpendicular to ray BC so that E and A are in the same half-plane determined by $\overleftrightarrow{BC}$ and BE = BA. Observe that ray BE is in the interior of angle ABC. Construct ray $\overleftrightarrow{CD}$ perpendicular to ray CB so D and A are in the same half-plane determined by $\overleftrightarrow{BC}$ and CD = BE.

Suppose the perpendicular bisectors of segments AD and BC interesect at O. Construct segments OA, OB, OC, OD and observe that OB = OC and OA = OD (Why?).

It follows that $\triangle OAB \cong \triangle ODC$ (Why?) and $\measuredangle OBA \cong \measuredangle OCD$.

So, $m\measuredangle ABC + m\measuredangle CBO = m\measuredangle DCB + m\measuredangle BCO$ and since $\measuredangle CBO \cong \measuredangle BCO$, we have $m\measuredangle ABC = m\measuredangle DCB = 90$; it follows that angle ABC is a right angle.

2.3.3 TRIANGLE INCONGRUENCE

In section 2.3.2, we explored triangle congruence criteria in the Euclidean model and Spherical model. We saw that while several of the congruence criteria involving sides and angles of triangles applied to both models, some did not.

Just as the notions of segment congruence and angle congruence, i.e., equality of the numbers associated with distance and angle measure, are critical in developing triangle congruence criteria, there are equally important notions predicated on the assumption that corresponding sides and angles of triangles are not congruent. We will begin the discussion of this most interesting topic by exploring relationships between sides and angles in a single triangle.

Consider triangle ABC in Figure 2.3v.

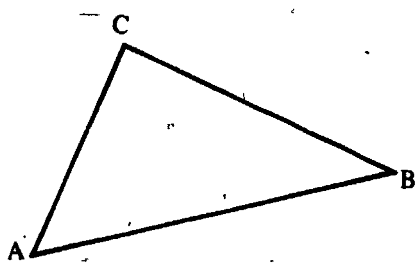

Figure 2.3v

Measure the lengths of sides AB, BC, and AC (to the nearest centimeter) and compare each of them to the sum of the lengths of any two of them. That is, compare the length of, say, side AC with the sum of the lengths of the other two sides, AB + BC. We see

$$AB + BC = 5cm + 4cm = 9cm > 3cm = AC$$

Repeating for the other two sides, we would see

$$AB + AC = 5cm + 3cm = 8cm > 4cm = BC$$
$$BC + AC = 4cm + 3cm = 7cm > 5cm = AB$$

This illustrates a very important mathematical principle referred to as the **Triangle Inequality**.

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In any triangle, the sum of the lengths of any two sides is greater than the length of the third side.

We really made no mention of whether this illustration was being made in the Euclidean model or Spherical model. Do you think it makes a difference?

Now let's examine the measures of the angles of triangle PQR in Figure 2.3w and their relationships with the sides of the triangle.

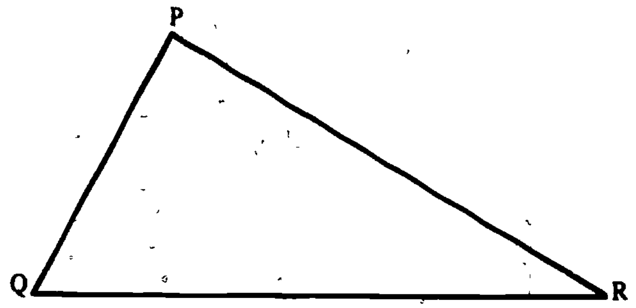


Figure 2.3w

Using a protractor, we find that:

$$m\measuredangle P = 88^\circ$$
$$m\measuredangle Q = 62^\circ$$
$$m\measuredangle R = 30^\circ$$

and using a ruler we find that:

$$QR = 8 \text{ cm}$$
$$PR = 7 \text{ cm}$$
$$PQ = 4 \text{ cm}$$

Angle P is the angle having the largest measure in the triangle and side QR is the longest side. It is not simply coincidence that side QR is opposite angle P. As a matter of fact, if we compare any of the measures of the angles of triangle PQR we see that the lengths of the sides opposite share the same relationship. This is summarized in the following table.

Angle measure comparisons	Length of side opposite comparisons
$m\measuredangle P > m\measuredangle Q$	$QR > PR$
$m\measuredangle P > m\measuredangle R$	$QR > PQ$
$m\measuredangle Q > m\measuredangle R$	$PR > PQ$

Thus, we have illustrated another interesting mathematical relationship.

If the measure of one angle of a triangle is greater than the measure of a second angle, then the length of the side opposite the first angle is greater than the length of the side opposite the second angle.

What about this statement when interpreted in the Spherical model?

Before leaving this example, let's interchange the notions of angle measure and segment length. That is, we would have the statement:

If the length of one side of a triangle is greater than the length of a second side, then the measure of the angle opposite the first side is greater than the measure of the angle opposite the second side.

Using Figure 2.3w, we could complete the following table to illustrate it.

Length of side comparisons	Angle measure opposite comparisons
QR > PR	$m\measuredangle P > m\measuredangle Q$
QR > PQ	$m\measuredangle P > m\measuredangle R$
PR > PQ	$m\measuredangle Q > m\measuredangle R$

To this point, interpretations in either the Euclidean model or Spherical model are valid. However, we encounter a problem with a familiar statement when the interpretation is in the Spherical model. The statement is:

In a right triangle, the length of the side opposite the right angle (hypotenuse) is the longest side.

To see this, study triangle NEQ in Figure 2.3x where N is at a distance one-fourth of the

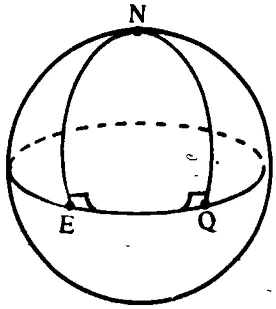


Figure 2.3x

circumference of a great circle from line EQ. We see that triangle NEQ has two right angles and the side opposite angle E, $\overline{NQ}$, is the same length as the side opposite angle Q, $\overline{NE}$. Also, the length of side EQ was chosen to be greater than the length of either of the other sides of the triangle. So, the statement is false since the longest side is not opposite the right angle of the triangle. As we have seen before, the problem resides in permitting N

to be at a distance of one-fourth the circumference of a great circle from line EQ. The statement will be interpreted in the Spherical model with this restriction added. That is,

> In the Spherical model if a right triangle has sides of length less than one-fourth the circumference of a great circle, then the length of the side opposite the right angle (hypotenuse) is the longest side.

The relationships so far have only addressed the sides and angles of one triangle. We are now ready to explore relationships involving two triangles. The first such relationship is the *SAS Incongruence Criteria* or *Hinge Theorem*:

> If two sides of one triangle are congruent respectively to two sides of a second triangle, and if the measure of the included angle of the first triangle is greater than the measure of the included angle of the second triangle, then the third side of the first triangle has length greater than the third side of the second triangle.

This situation is illustrated with triangle ABC and triangle DEF in Figure 2.3y, where $\overline{AB} \cong \overline{DE}$ and $\overline{AC} \cong \overline{DF}$ and $m\measuredangle A > m\measuredangle D$. Since $m\measuredangle A \neq m\measuredangle D$, triangle ABC is not congruent to triangle DEF under the correspondence ABC $\longleftrightarrow$ DEF. Measuring and comparing the lengths of segments BC and EF enables us to conclude BC > EF, thereby verifying the desired conclusion of the statement.

Figure 2.3y

We close this section with the *SSS Incongruence Criteria*, which is really the converse of the Hinge Theorem. Consider triangles ABC and DEF in Figure 2.3z where $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$ and BC > EF. How does $m\measuredangle A$ compare with $m\measuredangle D$? Using a protractor to

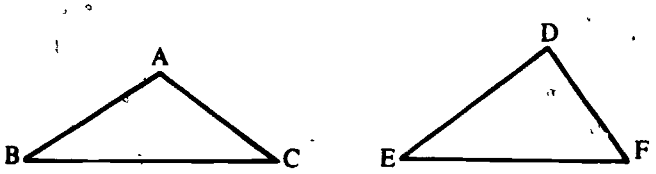


Figure 2.3z

measure angles A and D, we see that $m\measuredangle A > m\measuredangle D$. Thus, we are able to conclude a relationship between the *included* angles by knowing the relationship between the length of the sides opposite these angles. So,

> If two sides of one triangle are congruent respectively to two sides of a second triangle, and if the third side of the first triangle has length greater than the third

side of the second triangle, then the included angle of the first triangle has greater measure than the included angle of the second triangle.

Finally, can you formulate a statement that would probably be referred to as the ASA *Incongruence Criteria*, if it were true? This statement is:

If two angles of one triangle are congruent respectively to two angles of a second triangle, and if the included segment of the first triangle is longer than the included segment of the second triangle, then the measure of the third angle of the first triangle is greater than the measure of third angle of the second triangle.

An investigation of a few examples, both on the sphere and in the plane, will provide the reader with a better feeling for the validity or nonvalidity of the criteria. All too infrequently instruction in geometry focuses only on congruence relations and omits notions of incongruence in much the same way as inequality concepts are omitted from lessons concerning number concepts. There are several exercises at the end of this section that address this topic. Try them.

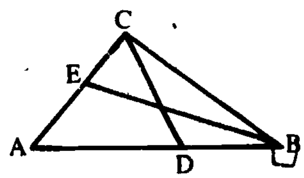
Exercises

1. Does the SAS *Incongruence Criteria* hold in both the Euclidean and Spherical models? How about the SSS *Incongruence Criteria*?

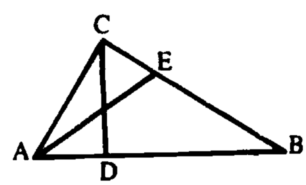
2. Formulate and examine statements involving all of the possible combinations of three sides/angles of two triangles to determine which of them yield valuable incongruence criteria.

In the remaining exercises in this section, all lengths are less than one-fourth of the circumference of a great circle if interpreting in the Spherical model.

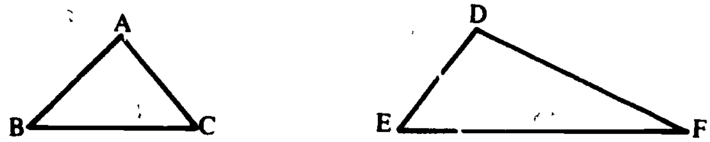
3. In the figure below, if AB > AC and EC = BD, then show CD < BE.



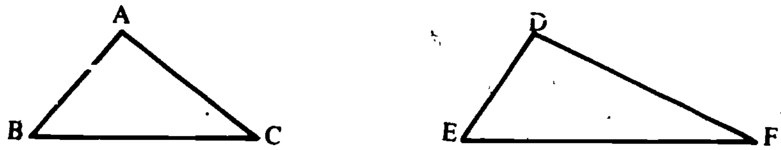
4. In the figure below, if CE = AD and CD < AE, then show BC < AB.



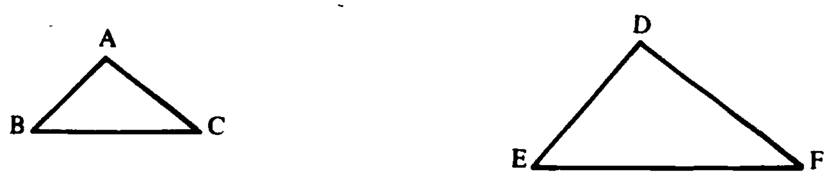
5. In the figure below, if $\measuredangle A \cong \measuredangle D$, $\overline{AB} \cong \overline{DE}$, and $m\measuredangle B < m\measuredangle E$, then show $m\measuredangle C > m\measuredangle F$.



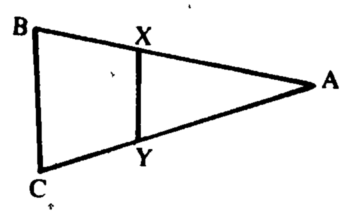
6. In the figure below, if $\overline{AB} \cong \overline{DE}$, $\measuredangle A \cong D$, and $AC < DF$, then show $m\measuredangle C > m\measuredangle F$.



7. In the figure below, if $\measuredangle A \cong \measuredangle D$, $\measuredangle B \cong \measuredangle E$, and $AB < DE$, then show $AC < DF$.



8. In the figure below, if $\overline{AB} \cong \overline{AC}$ and $\overline{AX} \cong \overline{AY}$, then show $XY < BC$.



2.4
Constructions

2.4.1 WHAT ARE THE BASIC CONSTRUCTIONS?

In this section we are concerned with simply introducing constructions in the spirit and tradition of the Greeks—constructions utilizing *only* a straightedge and compass. When one is actually able to perform these constructions, one possesses the tools for exploring some of the most interesting relationships among triangles and their associated circles. The review of the basic constructions below is intended as a refresher for those who have forgotten or as an introduction for those who have never been treated to this aspect of geometry before.

Construction 1

Copy a given line segment.
We are given segment AB and wish to copy it on the ray with endpoint X (Figure 2.4a).

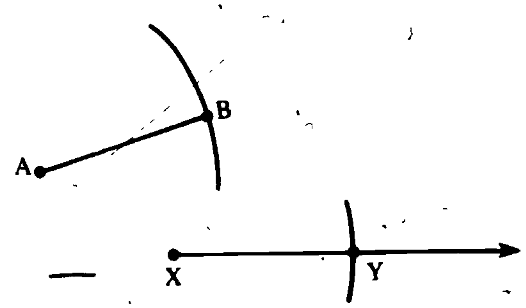


Figure 2.4a

We copy $\overline{AB}$ on this ray by constructing an arc of the circle centered at A having radius AB at X. The point where this arc intersects the ray, say Y, is the other endpoint of the desired segment. So, $\overline{AB} \cong \overline{XY}$. (Why?)

Construction 2

Copy a given angle.
We are given angle ABC and wish to copy it so that a side of the constructed angle is ray PS (Figure 2.4b).

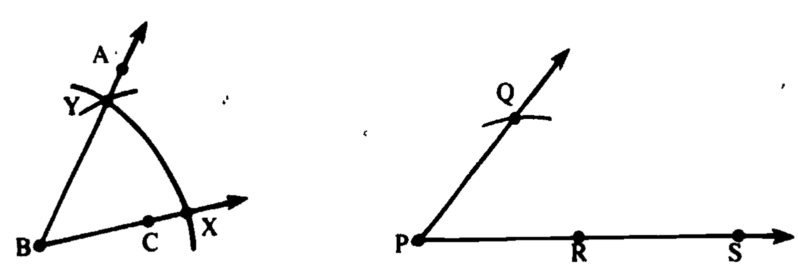


Figure 2.4b

Construct an arc of a circle having any radius centered at B and let X and Y be the points where this arc intersects the sides of angle ABC. Construct an arc of the circle having radius BX at P and designate the intersection of this arc with $\overrightarrow{PS}$ as R. Now construct an arc of the circle with radius XY and centered at R. This arc intersects the previously constructed arc centered at P; designate this intersection Q. Complete the construction by drawing ray PQ. Thus, angle QPR has been constructed and $\angle ABC \cong \angle QPR$. Why? (**Note:** It should be clear that since $\overleftrightarrow{PS}$ divides the plane into two half-planes, it is possible to repeat this construction and exhibit another angle having side PS congruent to angle ABC. We have addressed the construction in only one of the half-planes here.)

Construction 3

Copy a given triangle.

We are given triangle ABC and wish to copy one side of it on the ray with endpoint X (Figure 2.4c).

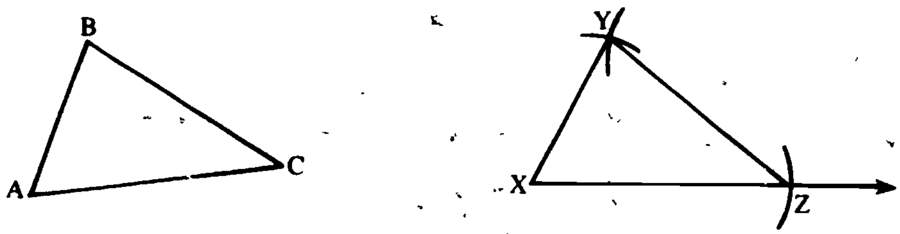


Figure 2.4c

First, construct an arc of the circle centered at A having radius AC at X; let Z be the point of intersection of the arc and the ray. Now, construct an arc of the circle centered at Z of radius BC. Also, construct an arc of the circle centered at X of radius AB. Denote the point of intersection of these arcs by Y. Now, complete the construction by drawing segments XY and YZ. Thus, we have constructed $\triangle XYZ$ and $\triangle ABC \cong \triangle XYZ$. (Why?)

Construction 4

Construct the perpendicular bisector of a given line segment.

We are given line segment AB and wish to determine a line that is both perpendicular to AB and passes through its midpoint (Figure 2.4d).

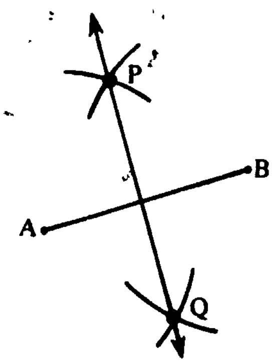


Figure 2.4d

Construct an arc of a circle centered at A having a radius greater than half the distance between A and B. (Why?) Using this same radius, construct an arc of the circle centered at B. Since the radius was suitably chosen, these two arcs will intersect at P and Q. Now, the construction is completed by drawing $\overleftrightarrow{PQ}$. Why is $\overleftrightarrow{PQ} \perp \overline{AB}$? Why is AM = MB?

Construction 5

Bisect a given line segment.

This construction is identical to Construction 4 where we recognize that M is the midpoint of line segment AB (Figure 2.4d).

Construction 6

Construct a perpendicular to a given line through a given point.

We are given a line l and a point P (not on l) and wish to construct a line on P that is perpendicular to l (Figure 2.4e).

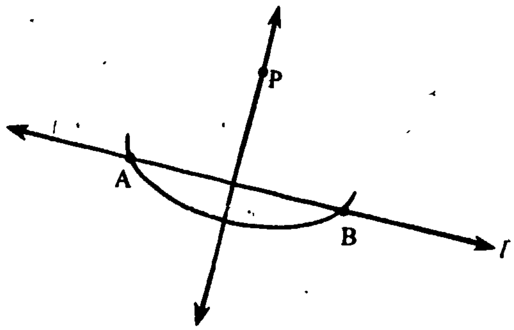


Figure 2.4e

Construct an arc of a circle centered at P having sufficiently large radius so that the arc intersects l in two points, A and B. The construction is essentially completed after recognizing that we can construct the perpendicular bisector of line segment AB (Construction 4). That is, we have constructed a line through P that is perpendicular to l.

It is important to observe that in our discussion the point P was *not* on l. How would one proceed if P were on l?

Construction 7

Construct the line that is parallel to a given line and contains a given point not on the given line.

We are given a line l and a point P not on l and wish to construct a line on P parallel to l (Figure 2.4f).

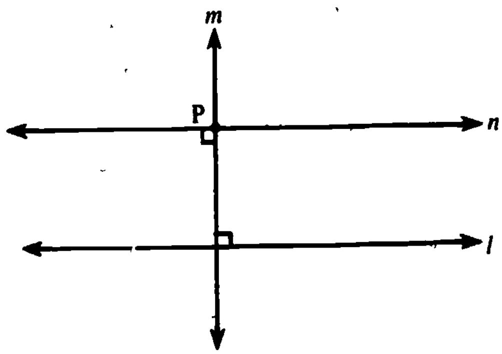


Figure 2.4f

Perhaps the simplest approach would be to repeat Construction 6 twice. First, construct line M perpendicular to l and containing P. Next, construct line n perpendicular to m and containing P. The line is thus parallel to l and contains P.

Can you provide an alternative construction that utilizes Construction 2?

Construction 8

Bisect an angle.

We are given angle ABC and wish to determine a ray with endpoint B lying in the interior of angle ABC and bisecting it (Figure 2.4g).

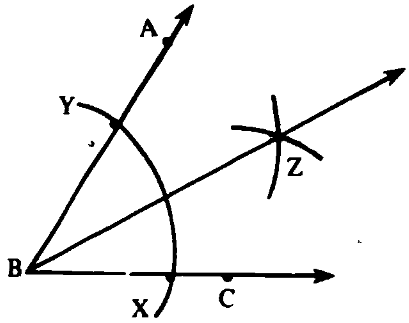


Figure 2.4g

First, construct an arc of a circle having any radius centered at B and let X and Y be the points where this arc intersects the sides of angle ABC. Now, with X and Y as centers, draw arcs of circles with radius BX (or any other radius for which these arcs will intersect) and designate the point of intersection in the interior of angle ABC as Z. Draw ray BZ, the desired bisector of angle ABC.

Construction 9

Construct a line segment on a given ray (containing the endpoint of the given ray) of length equal to n times the length of the given segment.

We are given line segment AB and we wish to construct a line segment on the ray with endpoint X that has length n times (n is a positive integer) the length of AB (Figure 2.4h).

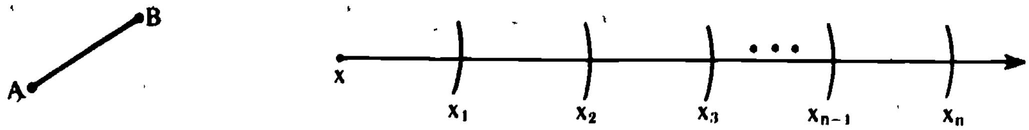


Figure 2.4h

We begin by copying line segment AB on the ray (Construction 1) and thereby determining X_1, so that $AB = XX_1$. Now, repeat Construction 1 using X_1 as the endpoint. That is, determine a point X_2 on the original ray so that $AB = X_1X_2$. So, n repetitions of Construction 1 carried out in this way will produce a line segment of length n times the length of line segment AB.

Construction 10

Subdivide a line segment into a given number of non-overlapping congruent line segments.

Suppose we have line segment AB and we wish to subdivide it into n non-overlapping congruent line segments (Figure 2.4i).

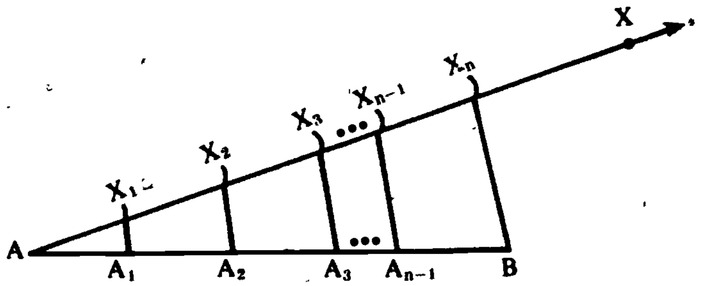


Figure 2.4i

We begin by drawing any ray having endpoint A and not containing B, say $\overrightarrow{AX}$. Construction 9 enables us to determine points $X_1, X_2, \ldots, X_n$ on ray AX so that $AX_1 = X_1X_2 = \ldots = X_{n-1}X_n$. Now, draw line segment X_nB. Using Construction 7, construct a line parallel to $\overline{X_nB}$ through X_1 and designate the point where this line intersects line segment AB by A_1. Repeat Construction 7 in this way at $X_2, X_3, \ldots, X_{n-1}$, thereby determining points $A_2, A_3, \ldots, A_{n-1}$ on segment AB. The segment AB has therefore been subdivided into n non-overlapping congruent segments, $\overline{AA} \cong A_1A_2 \cong \ldots \cong A_{n-1}B$.

Can you suggest an alternative construction procedure that utilizes Construction 2?

Activities

The activities below reinforce the constructions in this section. Try them.

1. Draw a segment of length 3 cm. Using only a straightedge and a single setting of your compass, construct an equilateral triangle whose sides are 3 cm in length.

2. Construct an isosceles triangle whose base is 2 cm long and whose congruent sides are 3 cm long.

3. Draw two acute angles having unequal measure. Construct an angle whose measure is the sum of their measures.

4. Draw any triangle ABC. Construct another triangle XYZ so that its perimeter is twice that of the perimeter of triangle ABC.

5. Construct a parallelogram PQRS from any given triangle PQR

6. Given a segment of length 3 cm, construct a square with a diagonal of length 3 cm.

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7. Construct a rhombus (parallelogram with sides of equal length), given the lengths of the diagonals.

8. Construct an angle whose measure is 45°.

9. Construct an angle whose measure is 60°.

10. Construct an angle whose measure is 105°.

11. Given a segment of unit length, construct a segment of length (a) $\sqrt{2}$, (b) $\sqrt{3}$, (c) $\sqrt{5}$.

12. Construct an isosceles right triangle whose longest side has length 4 cm.

13. Construct an equilateral triangle whose altitude has length 4 cm.

14. For any segment AB construct a segment whose length is $\sqrt{2}\,AB + \sqrt{3}\,AB$.

15. Describe how you would construct a segment whose length is k times the length of the given segment where k is a positive rational number.

16. Using exercise 15 and any segment AB, construct a segment whose length is $^2/_3$ AB.

2.4.2 EXPLORING TRIANGLE-CIRCLE RELATIONSHIPS

All too frequently construction activities degenerate into meaningless tasks with no real mathematical importance. The typical pattern for a unit on constructions requires that the student learn some basic construction techniques as discussed in the preceding section, practice each ad nauseam, make some artistic designs, and then forget them and return to the usual course concepts. We feel that techniques and applications of constructions can be integrated with course concepts to "discover" mathematical relationships.

This section deals specifically with mathematical relationships between a triangle and its associated circles and lines. The discovery format is followed and there are no proofs of any of the examples shown. We hope these activities will motivate you to develop specific activities appropriate to your teaching level.

Angle Bisectors

(1) Consider any triangle ABC and construct the angle bisector of each angle of the triangle (Figure 2.4j). Observe that these three angle bisectors are concurrent at a point referred to as the *incenter* of the triangle, and denoted I.

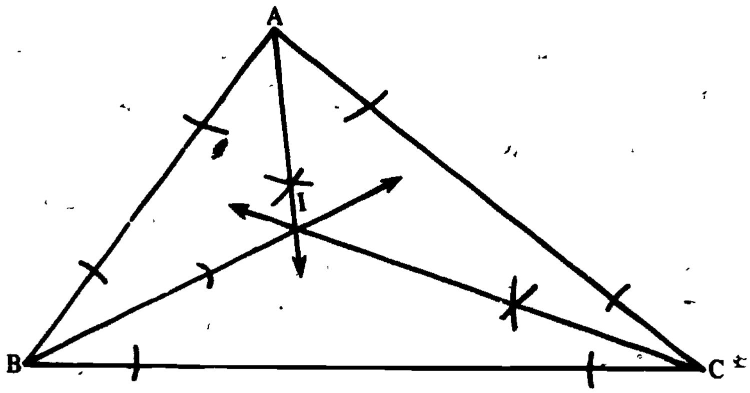

Figure 2.4j

Recalling an important property of an angle bisector, namely, that points on the angle bisector are equidistant from the sides of the angle, we know that point I is equidistant from the sides of the triangle. Thus, it must be the center of a circle which is tangent to the sides of the triangle. This circle is called the *incircle* of the triangle.

Now with A′, B′, C′ denoting the points of tangency of the incircle with $\overline{BC}$, $\overline{AC}$, $\overline{AB}$, respectively (Figure 2.4k), we consider $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$ and observe that these segments are concurrent at a point, usually referred to as the *Gergonne point* of the triangle and denoted G.

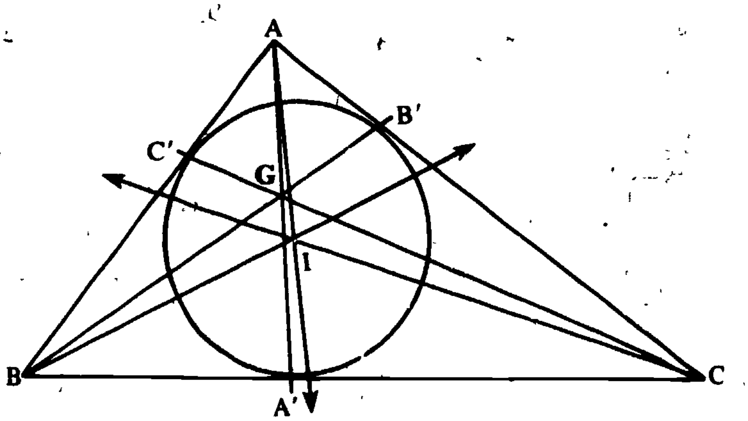

Figure 2.4k

(2) We now consider triangle ABC with its extended sides. First, we wish to determine the center of a circle, called an *excircle*, which is tangent to $\overline{AB}$, $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$ (Figure 2.4l). This problem in itself provokes some thought, and one possible solution requires the student to recall the property of angle bisectors mentioned in (1) above. With this in mind, it is clear, then, that the intersection of the bisectors of the appropriate exterior angles will determine the center, I_c, of the desired excircle and, hence, the excircle itself. This excircle is tangent to $\overline{AB}$ at a point, say C″.

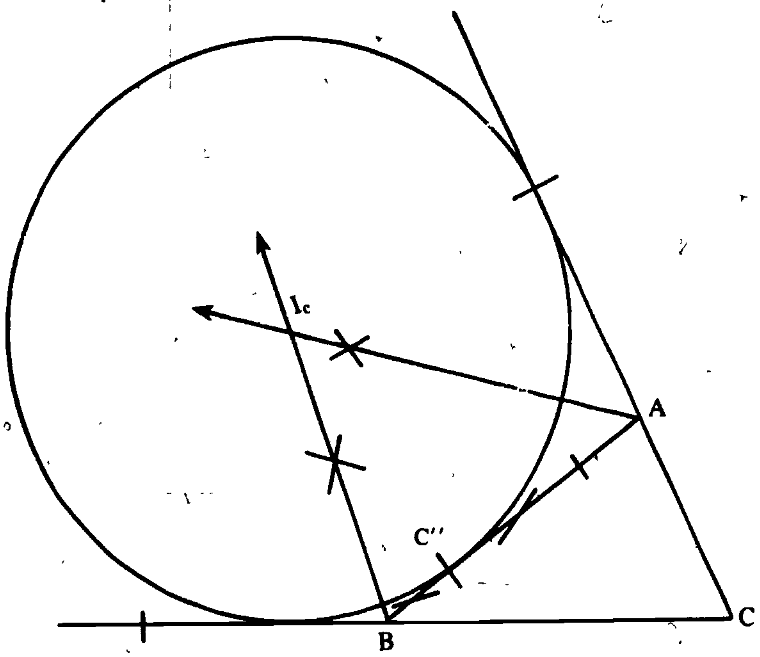

Figure 2.4l

Based on the procedure described above, it is possible to construct two more excircles, one tangent to $\overline{AC}$, $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$, while the other is tangent to $\overline{BC}$, $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$, with centers at I_b and I_a, respectively. We should note, however, that it is simple to minimize the number of constructions by using vertical angles. In light of this, how many angle bisectors must be constructed to locate the centers I_a, I_b and I_c of the desired excircles?

With the above excircles constructed with A″; B″ and C″ the respective points of tangency of the excircles to sides $\overline{BC}$, $\overline{AC}$ and $\overline{AB}$, we determine $\overline{AA''}$, $\overline{BB''}$ and $\overline{CC''}$ and observe that these segments are concurrent at a point, usually referred to as the Nagel point of the triangle and denoted **N** (Figure 2.4m).

The incircle described in (1) and the excircles described in (2) are called the four *Tritangent Circles* of the triangle; these circles are appropriately named as each is tangent to three sides of the triangle, or is tangent to one side and two extended sides of the triangle. These circles are related to still another circle associated with the triangle. These relationships we will explore later.

Two additional problems related to the angle bisectors of a triangle are:

(3) Determine if the points of intersection of the angle bisectors of two interior angles of a triangle and the sides opposite these angles are collinear with the point of intersection of the angle bisector of an exterior angle of the third vertex and the extended side opposite that vertex.

(4) Determine if the points of intersection of each of the bisectors of the exterior angles of a triangle and the extended sides opposite each vertex of the triangle are collinear.

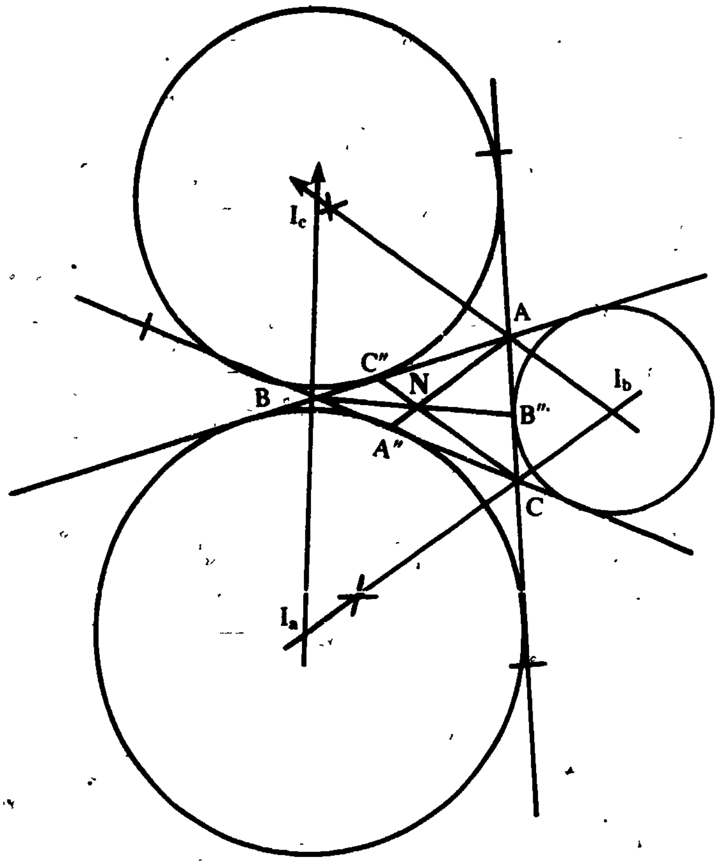

Figure 2.4m

Altitudes

(1) Consider any triangle ABC and construct the altitudes of the triangle (Figure 2.4n). We observe that the three altitudes are concurrent at a point, called the *orthocenter* and denoted H. It is also an interesting and constructive exercise to determine the location of the orthocenter for different types of triangles.

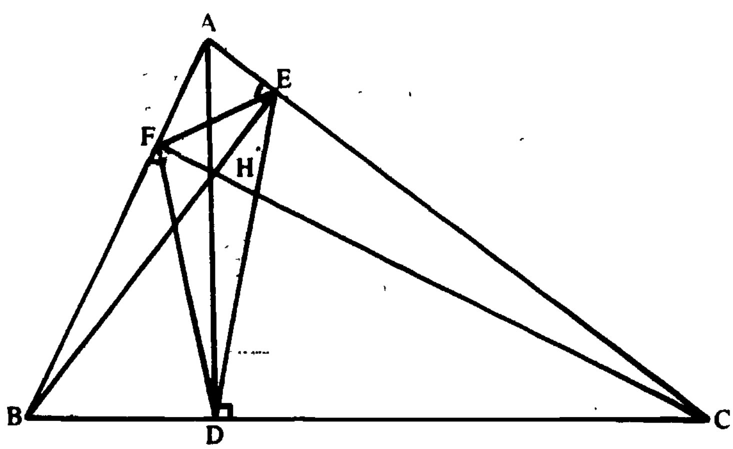

Figure 2.4n

(2) The feet of the altitudes, say D, E and F (Figure 2.4n), determine a triangle, called the *orthic triangle* of triangle ABC. There are many other interesting problems related to the orthic triangle:

(a) In Figure 2.4o, D and F are the feet of the altitudes from vertices A and C, respectively. Determine the relationship between $\overline{AC}$ and the perpendicular bisector of $\overline{DF}$.

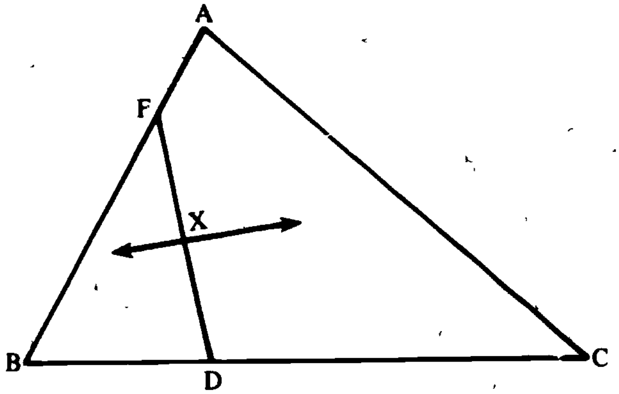

Figure 2.4o

Suppose we repeat the above for each of the remaining sides of the orthic triangle. Does a relationship exist among the three perpendicular bisectors so determined?

(b) In Figure 2.4n, construct the bisectors of the angles of the orthic triangle. What relationships did you find? From this construction may we conclude that the orthocenter of any triangle is the incenter of its orthic triangle? Be careful, note Figure 2.4p for an example when the given triangle is not acute.

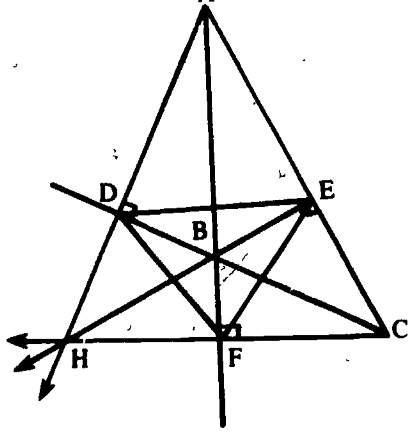

Figure 2.4p

(c) Fagnano's Theorem. In an acute-angled triangle ABC, the inscribed triangle (any triangle with its vertices on the interiors of $\overline{AC}$, $\overline{AB}$ and $\overline{BC}$) having minimal perimeter is the orthic triangle.

(3) Consider an equilateral triangle ABC (Figure 2.4q) and determine its incenter, I, and its orthocenter, H. What relationship exists between H and I? Construct the incircle and describe the points of intersection of the incircle with $\overline{AI}$, $\overline{BI}$ and $\overline{CI}$. Describe the orthic triangle of ABC. Does this concur with the results found in part (2)(b) above?

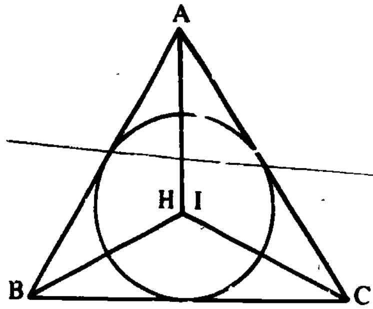

Figure 2.4q

Now, let P be any point on equilateral triangle ABC or any point in the interior of the triangle (some possible locations for P are given in Figure 2.4r), and find the sum of the distances from P to $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$. How does this compare with the length of the altitude?

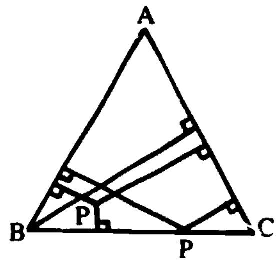

Figure 2.4r

(4) Returning to Figure 2.4m, construct the incircle, the center I, and label the points of tangency of the incircle to $\overline{BC}$, $\overline{AC}$ and $\overline{AB}$, A′, B′ and C′, respectively. Locate I_a, I_b and I_c, the centers of the excircles, and make observations for (Figure 2.4s):

(a) A, I and I_a

(b) B, I and I_b

(c) C, I and I_c

(d) triangles $I_aI_bI_c$ and A′B′C′

(e) $\overline{AI_a}$ and $\overline{I_cI_b}$

(f) $\overline{BI_b}$ and $\overline{I_aI_c}$

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(g) $\overline{CI_c}$ and $\overline{I_aI_b}$

(h) I and the orthocenter of triangle $I_aI_bI_c$

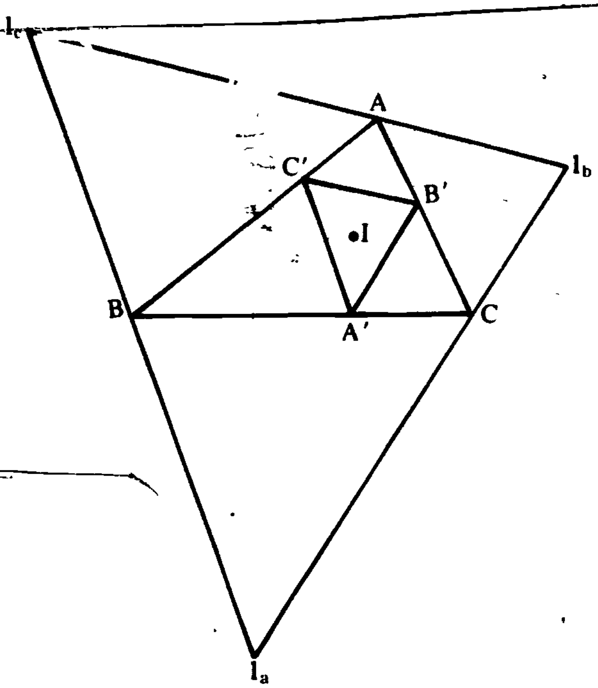

Figure 2.4s

Midpoints

(1) Consider triangle ABC with L, M and N, the respective midpoints of $\overline{BC}$, $\overline{AC}$ and $\overline{AB}$. $\overline{AL}$, $\overline{BM}$ and $\overline{CN}$ are called *Medians* of the triangle, and we observe that these three segments are concurrent in a point, usually referred to as the *Centroid* of the triangle and denoted G (Figure 2.4t). What relationship exists between AG and GL, BG and GM, CN and GN (consider AG/GL, BG/GM, CG/NG)?

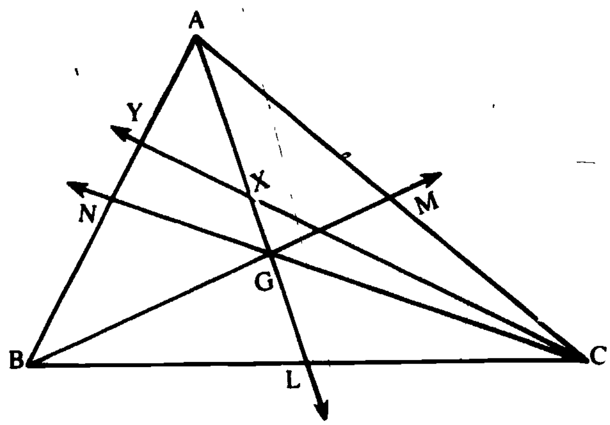

Figure 2.4t

In Figure 2.4t, let X be the midpoint of $\overline{AL}$ and $Y = \overrightarrow{CX} \cap \overline{BA}$. What relationship exists between distances AY and AB?

With the above remarks in mind, the following is left as a construction: Given three non-collinear points A, B, G, construct triangle ABC such that G is its centroid.

(2) Construct perpendiculars to $\overline{BC}$, $\overline{AC}$ and $\overline{AB}$ at L, M and N, respectively, and observe these lines (the perpendicular bisectors of the sides of the triangle) and concurrent at a point, called the *Circumcenter* of the triangle and denoted O. Since O is equidistant from A, B and C, it is the center of a circle containing A, B and C, called the *Circumcircle* of the triangle (Figure 2.4u).

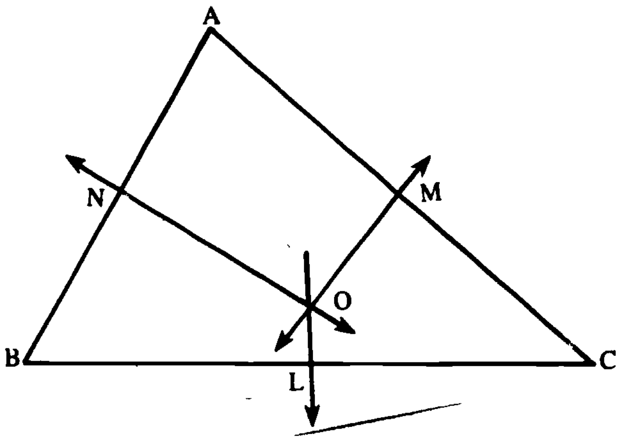

Figure 2.4u

(3) Some observations are now in order to consolidate portions of our findings.

(a) Consider a triangle ABC with H its orthocenter. Observe that each of the points A, B, C and H, is the orthocenter of the triangle formed by the other three points. Also, the circumcircles of the triangles ABC, BCH, CAH and ABH are congruent (Figure 2.4v).

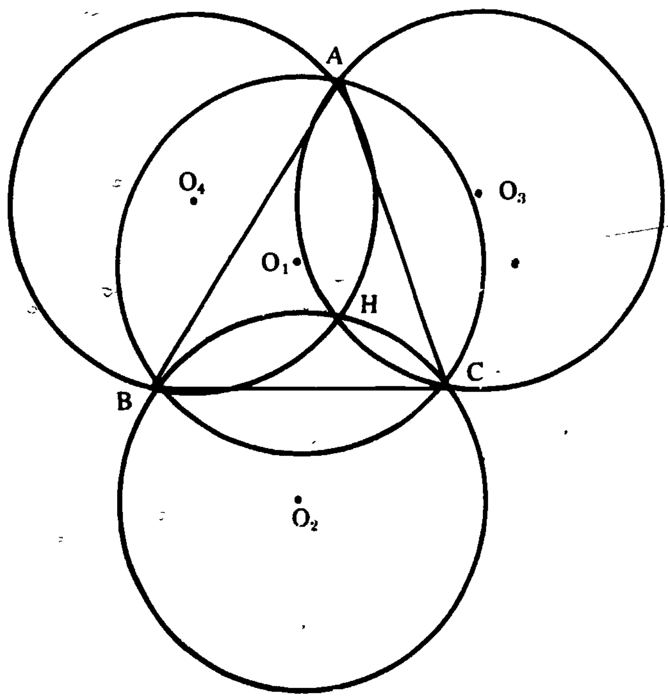

Figure 2.4v

(b) Let P be a point on the circumcircle of triangle ABC, and let J, Q and R be the feet of perpendiculars from P to BC, AC and BA, respectively. Observe that J, Q and R lie on a line, called the *Simson line* of P for the triangle (Figure 2.4w).

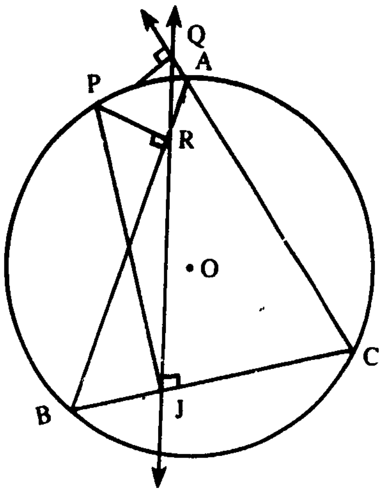

Figure 2.4w

(c) The triangle formed by the points of intersection of the lines containing the altitudes of the triangle with the circumcircle has its sides parallel to the corresponding sides of the orthic triangle of the given triangle (Figure 2.4x).

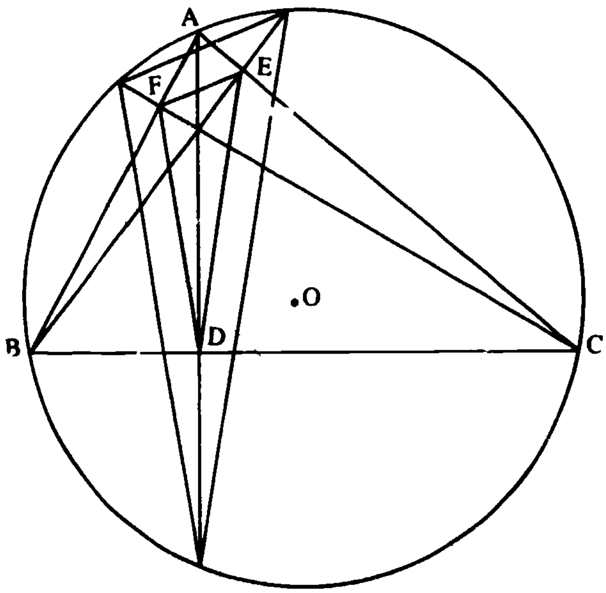

Figure 2.4x

(d) If the tangents drawn to the circumcircle at the vertices of the triangle intersect the opposite extended sides of the triangle, the points of intersection of the tangents with the opposite extended sides of the triangle are collinear.

(e) Consider triangle ABC with H its orthocenter, G its centroid, and O its circumcenter (Figure 2.4y). We observe that H, O and G lie on a line, called the *Euler line* of the triangle. Another interesting relationship exists among H, O and G; can you decide what it is? It is helpful to consider HO/OG.

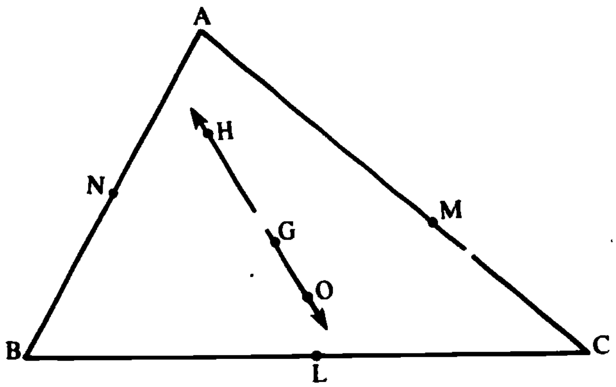

Figure 2.4y

Nine-Point Circle

Consider triangle ABC with H its orthocenter, and construct the following groups of points: (1) the midpoints of the sides of the triangle, say L, M and N; (2) the feet of the altitudes on the sides of the triangle, say D, E and F; (3) the midpoints of HA, HB and HC, say X, Y and Z, respectively. These points all lie on a circle, the *Nine-Point Circle*. (It was O. Terquem who named the circle the Nine-Point Circle, and this is commonly used in English-speaking countries. Some French geometers refer to it as Euler's Circle, and German geometers usually call it Feuerbach's Circle.) See Figure 2.4z below.

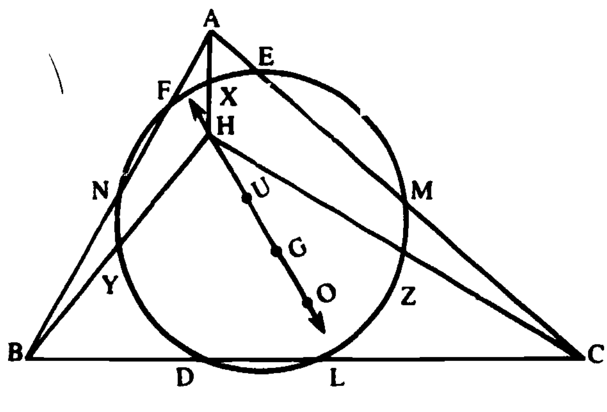

Figure 2.4z

There are many observations that can be made with regard to the nine-point circle, a few of which are listed below.

(a) The radius of the nine-point circle is one-half the radius of the circumcircle.

(b) If U denotes the center of the nine-point circle, then U is the midpoint of $\overline{HO}$, and hence, U is on the Euler line.

(c) The nine-point circle is the circumcircle of the orthic triangle.

(d) Tangents constructed to the nine-point circle at L, M and N, respectively, are parallel to their respective sides of the orthic triangle.

(e) Feuerbach's Theorem: The nine-point circle is tangent to each of the four tritangent circles of the triangle.

(f) If P is on the circumcircle, and J and Q are as in Figure 2.4w, then $\overline{PH} \cap \overleftrightarrow{JQ} = K$ is on the nine-point circle and PK = KH.

(g) Let T_1 be 1/3 of the distance from M to E, T_2 be 1/3 of the distance from N to F, and T_3 be 1/3 of the distance from L to D, then the triangle $T_1T_2T_3$ is equilateral.

(h) The circumcircle of triangle ABC is the nine-point circle of triangle $I_aI_bI_c$ where I_a, I_b, I_c are the centers of the excircles of triangle ABC.

(i) The midpoints of $\overline{I_cI_b}$, $\overline{I_bI_a}$, and $\overline{I_aI_c}$ lie on the circumcircle of triangle ABC.

(j) The orthcenter, O, of triangle ABC is the midpoint of the segment joining the incenter, I, of triangle ABC to the circumcenter of triangle $I_aI_bI_c$.

A Final Problem

Consider any triangle ABC. Construct an equilateral triangle ABC' with sides of length AB on side AB so that C' is *not* in the same half-plane as C determined by $\overleftrightarrow{AB}$. Repeat this construction using the other sides of triangle ABC. We have determined three equilateral triangles, named ABC', AB'C and A'BC (Figure 2.4z').

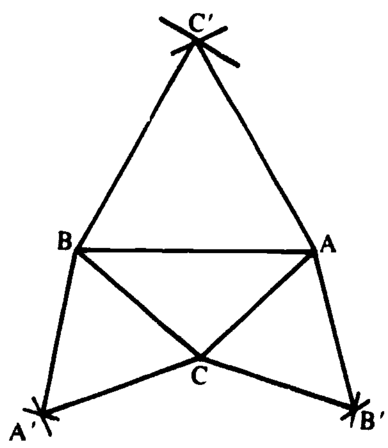

Figure 2.4z'

Considering $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$, we see they are concurrent at a point called the *Fermat point* of the triangle and denoted F. What conclusions can be reached about the lengths of these segments?

Suppose we permitted C′ to be in the same half-plane as C determined by $\overleftrightarrow{AB}$, and permitted B′ and A′ to satisfy similar conditions. What can now be said about $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$? About $\overline{AA'}$, $\overline{BB'}$ and $\overline{CC'}$?

The examples above are important as they relate to the geometry of the circle and triangle. Frequently, studies in geometry omit these relationships because of the difficulties that arise should one desire to formally prove each assertion. We believe that these are suitable for any lab situation where the purpose is to "discover" relationships, with the proof of such relationships left open. None of the above was intended to serve as a rigorous proof but only to imply how much more we could do with constructions which at the same time enhance the mathematics being learned by the students, even if the learning occurs intuitively.

We feel that the above are also important in that most of the geometric constructions occur—and combinations of many of them occur—in a single problem. It is our hope that the imaginative mathematics teacher can utilize these examples to design discovery lessons, math lab activities, or enrichment opportunities.

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2.5

Transformational Geometry

In this section we are going to look at a specific model of transformational geometry. There are three basic types of transformations which can be considered:

(a) Topological transformations, modeled as changes in shape and size;

(b) Rigid transformations, modeled as changes in position (the one we will look at further); and

(c) Projective transformations modeled as changes in visual perception.

Topological transformations are best shown by considering an inflated balloon and maneuvering it into different shapes. Topological transfc.mations are possible in the balloon as long as you do not cut it or fold two points together.

Projective transformation concepts are best shown by shadows. One concept uses parallel light rays to cast shadows (such as those cast by the sun) while another concept uses light rays which originate from a single point (a flashlight). Both topological transformations and projective transformations allow for changes in size and shape, while rigid transformations do not, and as such, rigid transformations appear easier to study because all they allow for are changes in location. In other words, rigid transformations deal with the concept of congruence (see section 2.3).

There are three basic kinds of rigid transformations that can successfully be studied in the school—*slides*, *flips*, and *turns*. We use this terminology because of its comprehension by students; mathematicians would use the words translations, reflections. and rotations. Before we discuss slides, flips, and turns, we make a few comments. All transformations have the effect of "acting" on an object to change it in some way. This prompts the thought of what the object looked like before it was changed and what it looks like after the change. Instead of using "before" and "after" to connotate these concepts we use the words "pre-image" and "image" to indicate the same idea.

2.5.1 WHAT IS A SLIDE?

First of all, let us consider slides. Two easy ways of considering slides are the shooting gallery and a children's slide illustrated in Figures 2.5a and 2.5b.

Using colored pencils, the reader should make a red S to indicate a starting location (pre-image) and a blue S to indicate where the red S ends up (image) after the slide transformation has been applied. It is well to note that slides do not have to be horizontal or vertical and that there are two important concepts connected with slides which can be seen by the use of a slide arrow. The arrow indicates a "direction" and a "length," and is not to be treated as a ray. While the concept of what to do may be readily seen, the actual drawing of the image, given the arrow and the pre-image, may prove more difficult. The following exercises will give you needed practice in finding images. We suggest you have a straightedge, compass, paper, and red and blue colored pencils ready to use for these exercises.

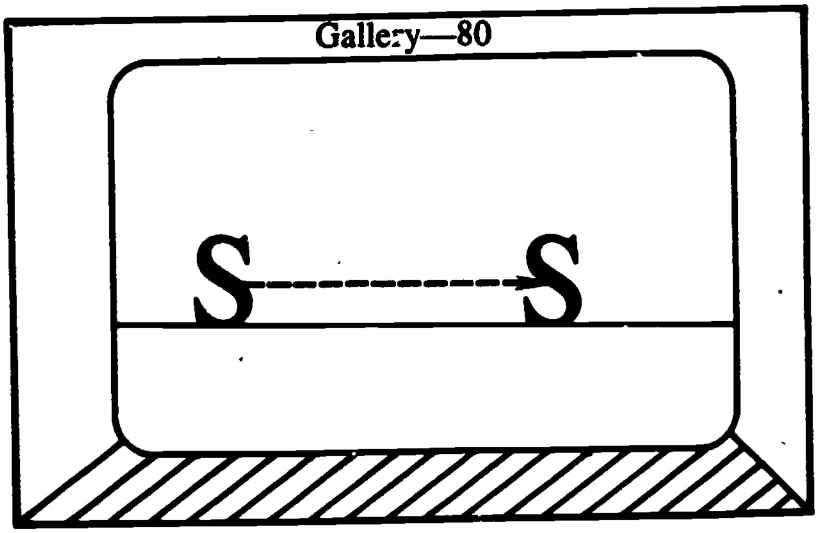


Figure 2.5a

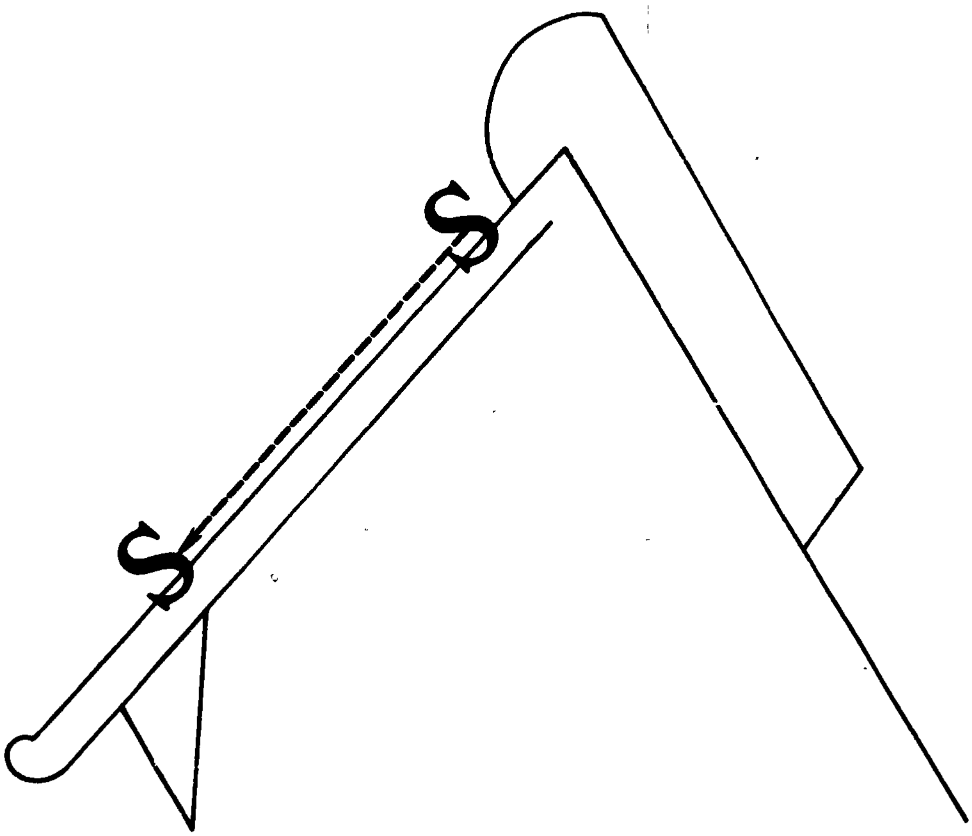


Figure 2.5b

Exercises

Draw the images of the following shapes, given the pre-image and the slide arrow.

(1)

(2)

(3)

In each of the preceding three exercises it should be noted that the pre-image and the image were congruent. In this case we become more specific and say the image and pre-image are "slide congruent."

2.5.2 WHAT IS A FLIP?

Second, let us look at flips. Flips can also be considered through another aspect of a shooting gallery (see Figure 2.5c) as well as through the use of a mirror. The use of a mirror has prompted some people to use the terminology "mirror images" when referring to flips, although the use of a mirror as a model for flips has limitations.

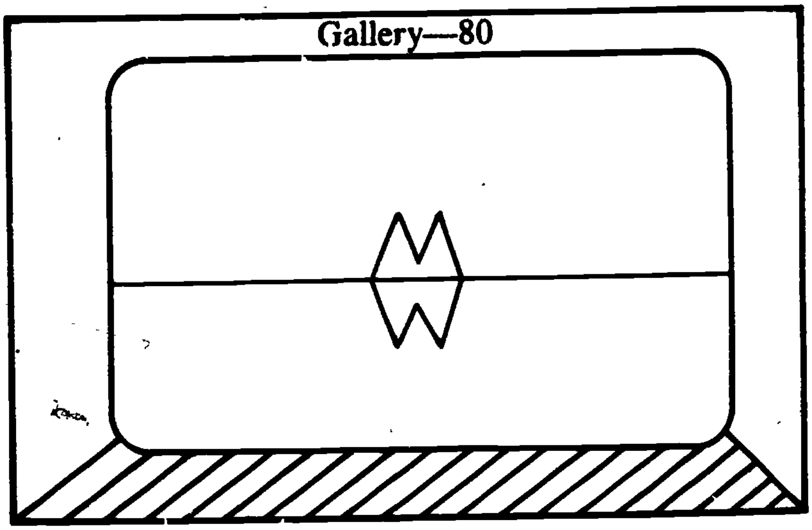


Figure 2.5c

Still using colored pencils, the reader should make a red M to indicate the starting location (pre-image) and a blue M (image) to indicate what would happen to the red M if it was the target of an expert shooter. Again, we note that there is an interesting feature associated with flips—there is always a line about which the flip is made. The Gallery 80 idea is a bit limiting in its location of the flip line so that we give the following examples (Figure 2.5d) also, using a solid line for pre-image and a dotted line for image.

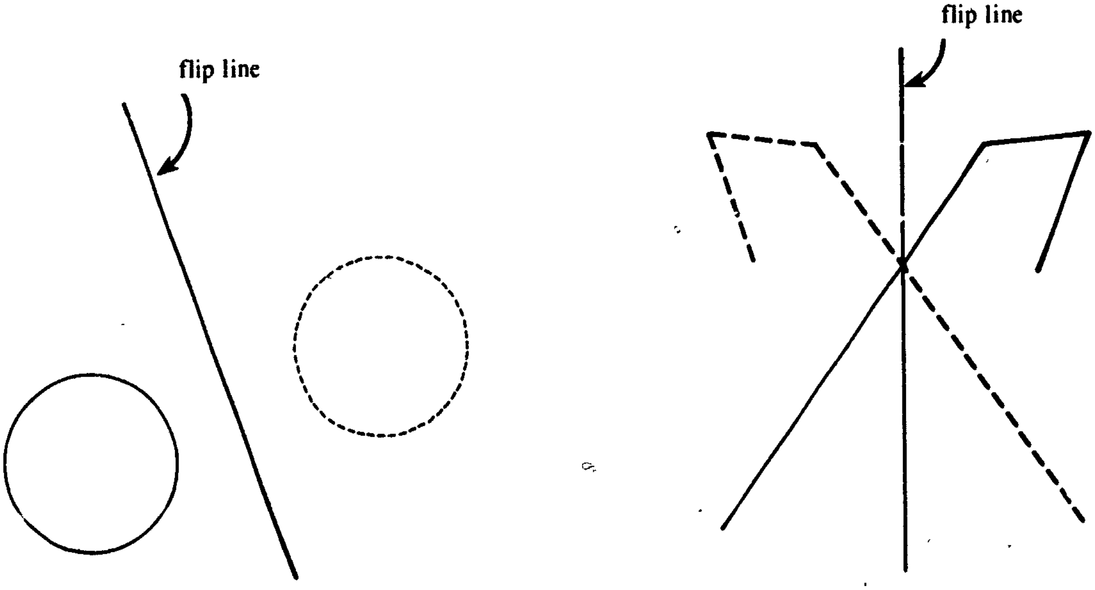


Figure 2.5d

The two examples indicate that (a) the image and pre-image may be on opposite sides of the flip line and not touching it; (b) the image and pre-image may both lie on both sides of the flip line; (c) if the pre-image touches the flip line, so will the image; and (d) the pre-image and image are congruent, as they were with slides. In case (d), we say the image and pre-image are flip congruent. The concepts of a flip line and flip images are usually easy to grasp; however, the actual construction of a flip image is not as easy to do. The following few exercises were selected to encourage you to participate in the construction of flip images.

Exercises

With the above examples and considerations in mind, complete the following.

1. For the flip line properly chosen (Figure 2.5e), the letter T is both the pre-image and image (in which case we say that T is the image of itself). What other letters of the alphabet can be made into images of themselves by an appropriate choice of a flip line? Note that the style of printing may make a difference as to whether or not a letter can be made into an image of itself by an appropriate choice of flip line.

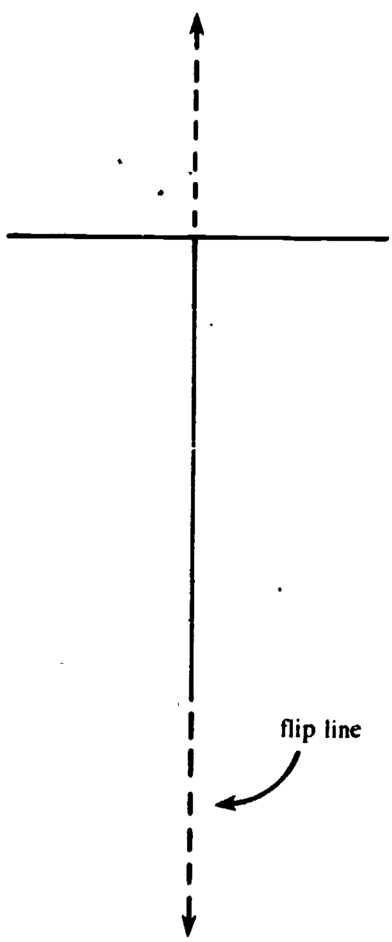
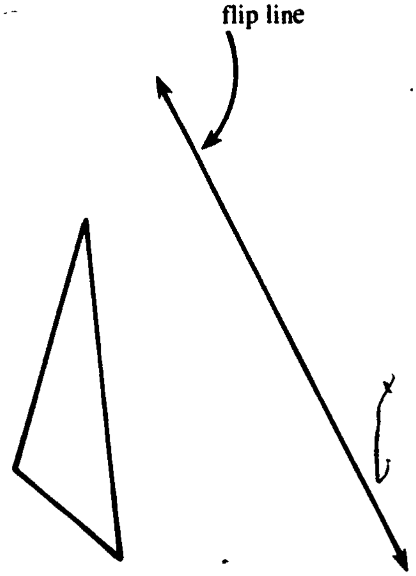


Figure 2.5e

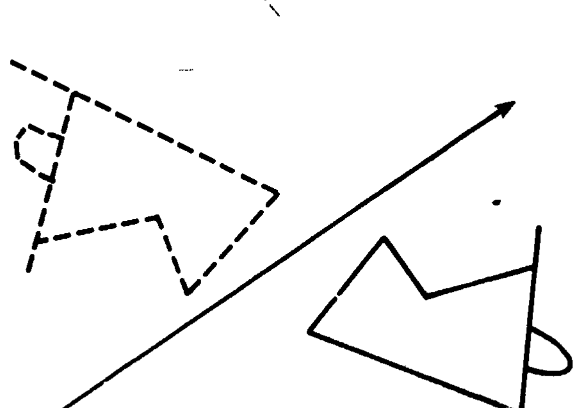
Note: This exercise leads to a concept worth noting, but upon which we choose not to dwell. The concept is symmetry. Symmetry is dealt with in many ways, especially in art, in middle schools, and the particular symmetry noted above is called line symmetry with the flip line being called "the line of symmetry." Any geometric figure has a line of symmetry if it is the image of itself using some flip line. For many examples of line symmetry, consider any middle school textbook series, the article by Sanok, or carefully look at "Mother Nature."

2. Using the given flip line and triangle as the pre-image, find the image.



3. Using the given flip line and pre-image, complete the image. Can you find other words that provide similar happenings (try a vertical flip line through the O)?

4. Is the dotted figure the image of the solid figure under the given flip? Why? How can you easily tell?



Note that in the fourth exercise, although the figures are congruent, they are not flip congruent!

At this point, the wise teacher will make use of the concept of folding or tracing to: (a) find the flip line if the pre-image and image are given, or (b) find the image given the pre-image and the flip line. We encourage the teacher to indicate that flip lines are unique when found in (a), and flip images are unique when found in (b).

2.5.3 WHAT IS A TURN?

The third transformation we wish to introduce is that of a turn, and once again we return to the shooting gallery (Figure 2.5f), but introduce a different picture. Here we use the "spinning" target idea. We note that a model of this can be easily constructed with a paper plate and a brad fastener.

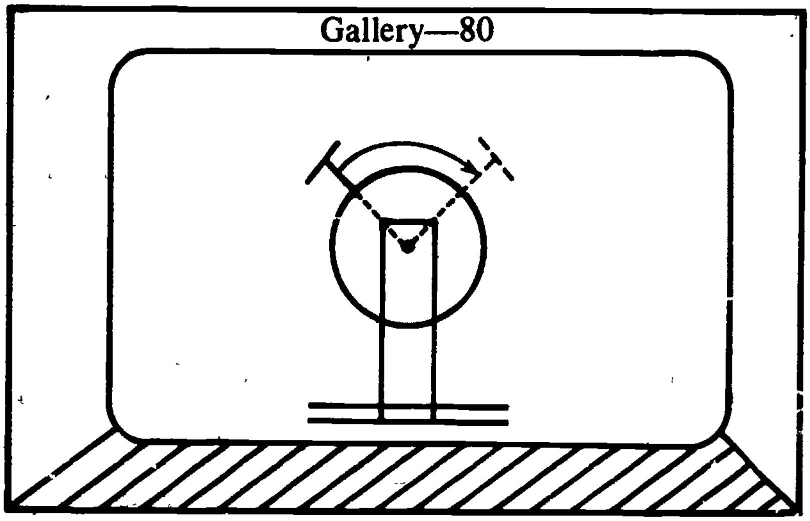


Figure 2.5f

From this idea, consider the solid T as the starting location (pre-image) and the dotted T as the ending location (image) and note the following about a turn: (a) there is a center for the turn (some call it a "pivot"), (b) there is a curved arrow that is associated with a turn. This arrow shows both the direction and amount of turn, (c) if we connect the ends of the arrow to the center of the turn, an angle is formed which can be associated with the turn, and (d) the image and pre-image are congruent, and we use the term "turn congruent."

There are other examples that can help picture the concept of a turn. For the following examples, try to sketch a picture for the turn indicated. Also, locate the center of the turn and the turn arrow. Consider:

(1) an end view of a swing with a person swinging,

(2) a scissors with one side stationary,

(3) a ferris wheel,

(4) a person on a trapeze,

(5) a pendulum on a clock,

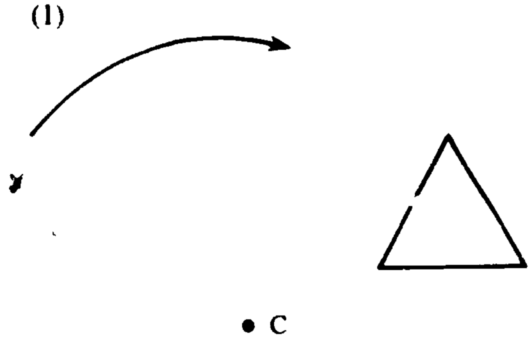
(6) a hand on a clock, and

(7) consider others not mentioned above.

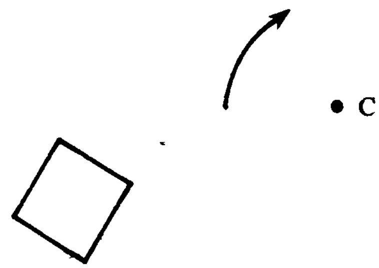
Exercises

Upon completion of the above, try the following. In exercises 1–4, you are given the pre-image and turn arrow with center C; find the image. You may wish to use "onion skin" paper to do these exercises. If you place two sheets of it on top of each other, you can physically turn one sheet the proper amount and you will be able to properly see the correct image.

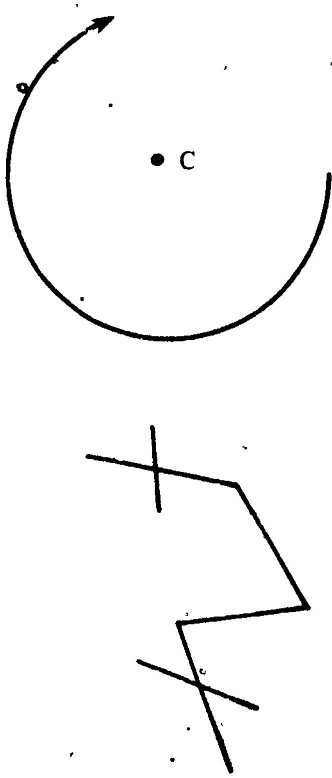
(1)



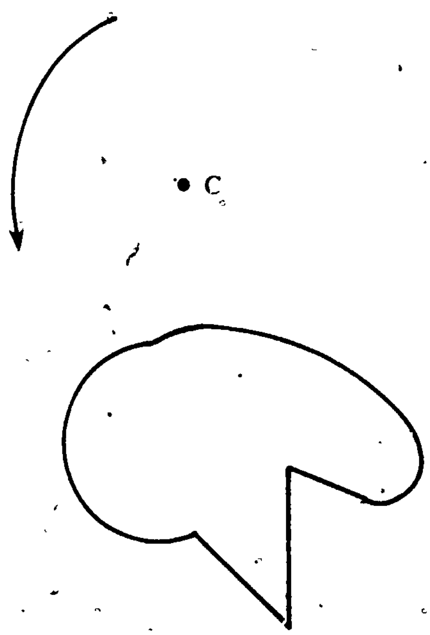
(2)



(3)

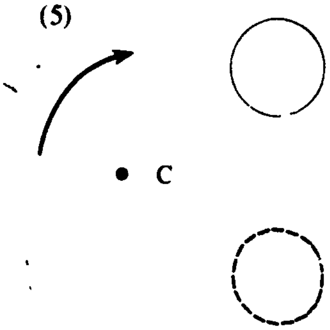


(4)

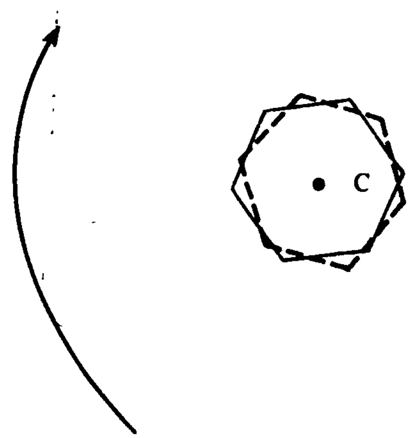


In exercises 5–8, determine if the dotted figure is the image of the solid figure under the given turn with center C and turn arrow specified. You again may wish to operate in a manner similar to that suggested in exercises 1–4.

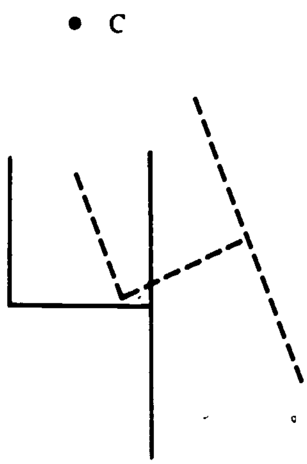
(5)



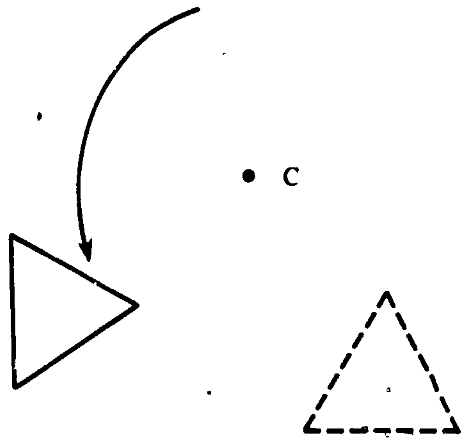
(6)



(7)



(8)



2.5.4 EXPLORING SUCCESSIVE MOTIONS

We are now going to examine the situation where we do one rigid transformation and follow it by another rigid transformation. We are going to restrict ourselves to an extensive study using slides and turns, while leaving open many other combinations which are suggested in the activities.

Successive Motions of Slides

Suppcse we have two slide arrows, labeled S_1 and S_2 as in Figure 2.5g. Finding the slide image of P using slide arrow S_1 is done by using the solid arrow.

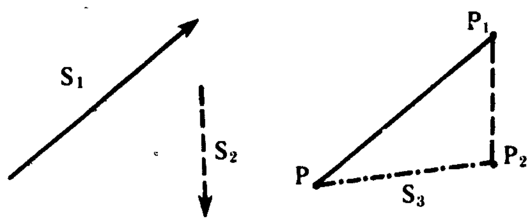


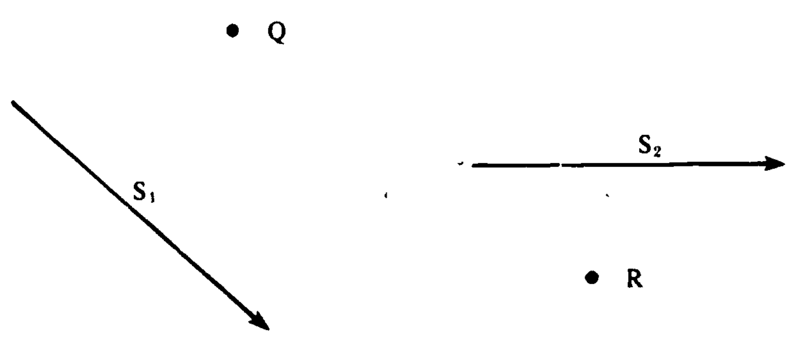
Figure 2.5g

Using arrow `$_1$, if we call P_1 the slide image of P, we can find the slide image of P_1 under arrow S_2—which is done by using the dotted arrow. If we call this point P_2, we say P_2 is the image of P, and denote that by $S_2S_1(P) = P_3$, we can also readily discern an arrow S_3 that makes P_2 the image of P, and write $S_3(P) = S_2S_1(P)$.

Exercises

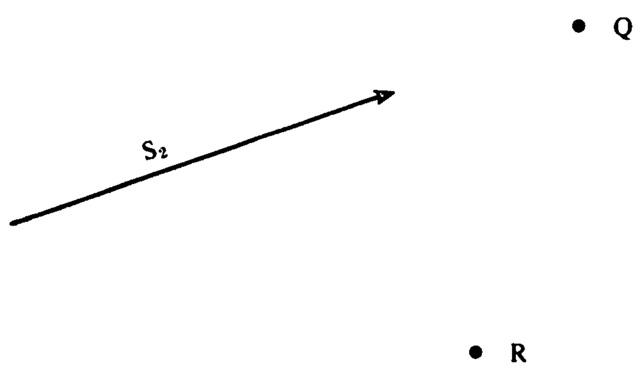
1. (a) In figure 2.5g, use the slide arrows, S_1 and S_2 and point P to find $S_1S_2(P)$. (Note that this means we first use slide arrow S_2 and then slide arrow S_1.)
   (b) After having done $S_1S_2(P)$ and $S_2S_1(P)$, what kind of figure is formed? What relationship does $\overline{PP_2}$ have to the figure formed?

2. (a) Find $S_1S_2(Q)$ and $S_1S_2(R)$ for S_1, S_2, Q and R given below, calling $S_1S_2(Q) = Q_2$ and $S_1S_2(R) = R_2$.

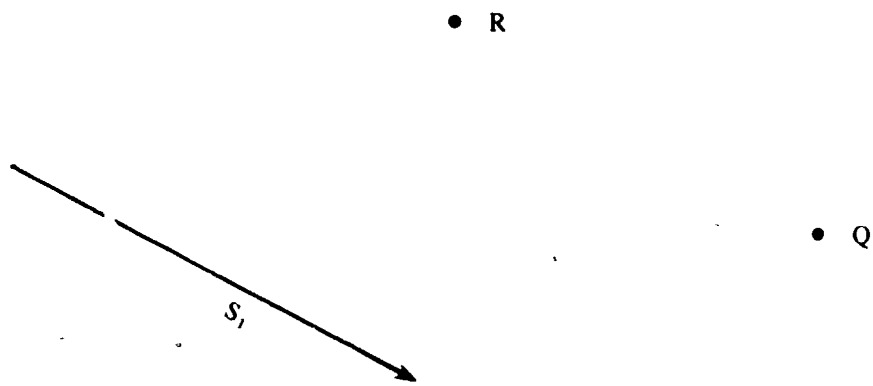


   (b) Measure $\overline{QQ_2}$ and $\overline{RR_2}$. What do you notice?

3. Find S_1 such that $S_1S_2(Q) = R$ below.



4. Find S_2 such that $S_1S_2(Q) = R$ below.



Note that in exercise 1(b), a parallelogram is formed with $\overline{PP_2}$ as a diagonal. In exercise 2(b), $\overline{QQ_2}$ and $\overline{RR_2}$ have the same length and are parallel. Exercises 3 and 4 provide opportunities to help visualize what is actually happening with slides and also points out the necessity to think about "inverse" relations. You will know the solution when you find it.

In our previous discussion we were very narrow in our choices for S_1 and S_2. Further investigations are warranted by choosing S_1 and S_2 to be "parallel" or to be going "opposite" directions In fact, using S_1 and S_2, going in the same or opposite directions is precisely the model that is most often used on the number line to "pictorially" depict addition and subtraction of whole numbers and integers. It is with integers that arrows first receive attention as portraying a situation that has two variables--"direction" and "distance." It is to this end that the above exercises are aimed.

It should also be noted that you can do successive motion of slides with any number of slide arrows.

Successive Motions of Turns (with the Same Center)

Analogous to what was presented above with slides, we want to consider what happens if we do successive motions of two turns with the same center.

Suppose we have two turn arrows, T_1 and T_2, at center C (Figure 2.5h). Find the turn image of P (calling it P_1) using T_1. Now, if we find the turn image of P_1 (call it P_2) using

Turn T_2, we say that P_2 is the image of P and we denote that by $T_2T_1(P) = P_2$. We also can discern two turn arrows, T_3 and T_4, that make P_2 the turn image of P, and write $T_3(P) = T_4(P) = T_2T_1(P)$.

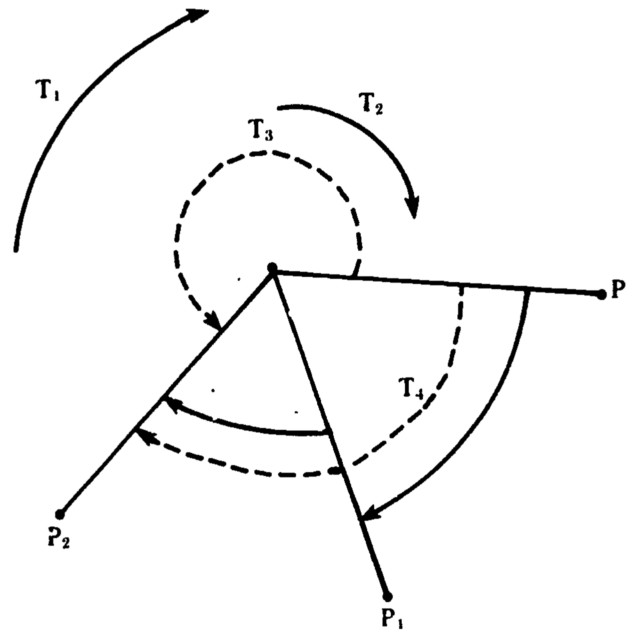
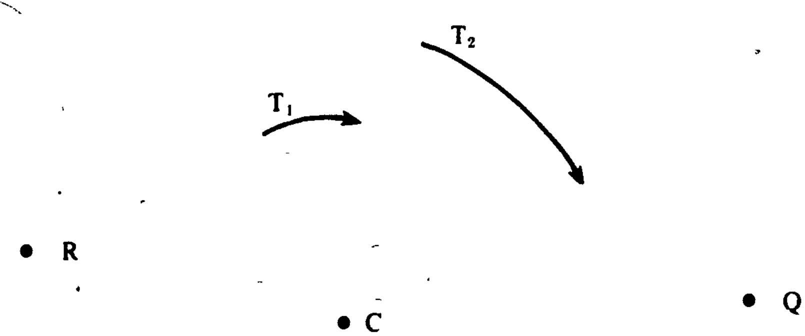


Figure 2.5h

We can see that $T_4(P) = P_2$, as well as $T_3(P) = P_2$. At this stage, we are going to allow both possibilities.

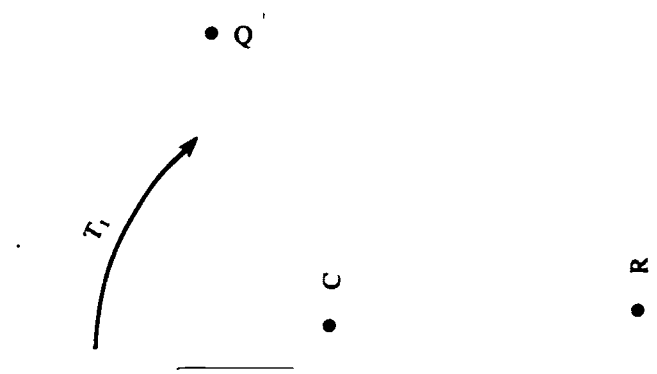
Exercises

1. Using the above turn arrows (Figure 2.5h), T_1 and T_2 and point P, find $T_1T_2(P)$. (Note that this means that first we use turn arrow T_2 and then turn arrow T_1). What conjecture can you make?
2. (a) Using points Q and R and turn arrows T_1 and T_2 with center C below, find $T_1T_2(Q) = Q_2$ and $T_1T_2(R) = R_2$.

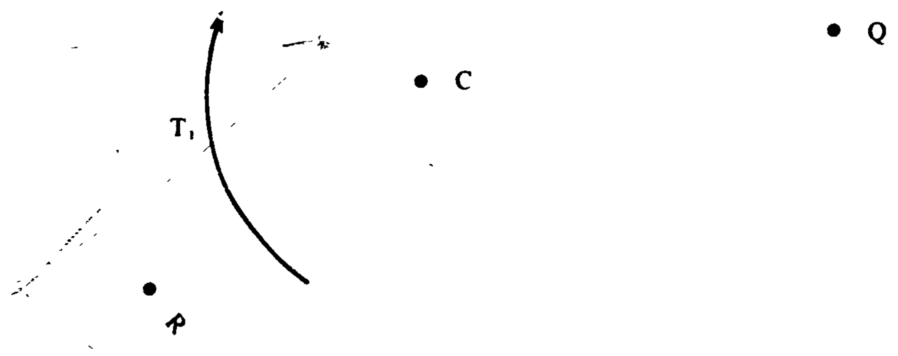


(b) Measure angles QCQ_2 and RCR_2. What do you notice?

3. (a) Find T_2 such that $T_2T_1(Q) = R$ below (using C as the center for both turn arrows).



(b) Find T_2 such that $T_1T_2(Q) = R$ below (using C as the center for both turn arrows).



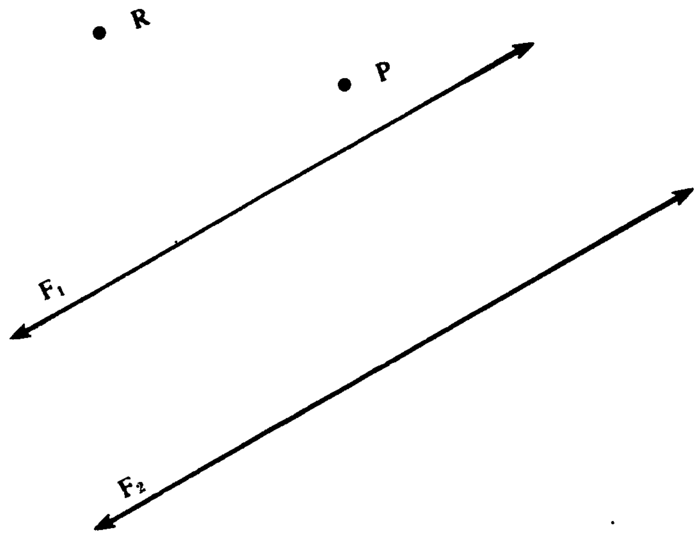
Note: In exercise 1, $T_1T_2(P) = T_2T_1(P)$; thus order is not important if turns are around the same center. In exercise 2(b), the measure of the angles is the same.

Activities

1. For exercises 3(a) and 3(b) above, you should be able to locate two possibilities for turn arrow T_2. Also, as with slides, we have been very narrow in our choices for T_1 and T_2 in that we have had them go the "same direction" and they have the same center. A situation similar to slides occurs if the turns have the same center but "opposite directions," but a more interesting situation arises if T_1 and T_2 have different centers, and is well worth the investigation, but it is beyond the scope of this book.

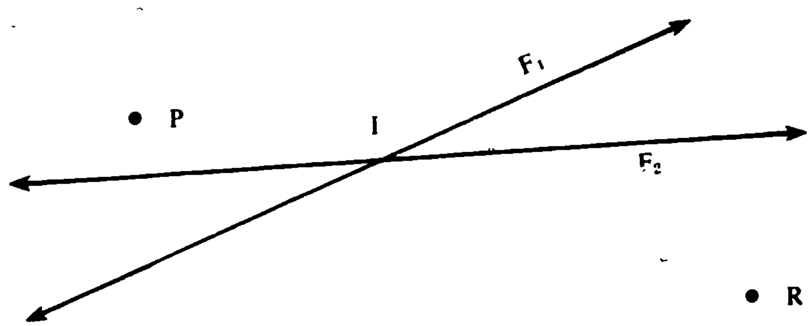
2. For more advanced students, investigations in the following areas can be used. It should be noted that these investigations are just as important as the ones we have provided above, but are more complicated.

   (a) If we let F_1 and F_2 be two parallel flip lines, find $F_1F_2(P) = Q_1$, $F_2F_1(P) = Q_2$, $F_1F_2(R) = V_1$ and $F_2F_1(R) = V_2$.



Measure $\overline{PQ_1}$, $\overline{PQ_2}$, $\overline{RV_1}$ and $\overline{RV_2}$, and what do you notice? We note that Q_1 and Q_2 are different points, as are V_1 and V_2, which is different from what happened in prior exercises concerning successive motions with slides and with turns using the same center. Furthermore, F_1F_2 looks like a slide as does F_2F_1. Try more cases for F_1 and F_2, P and R and see what happens!

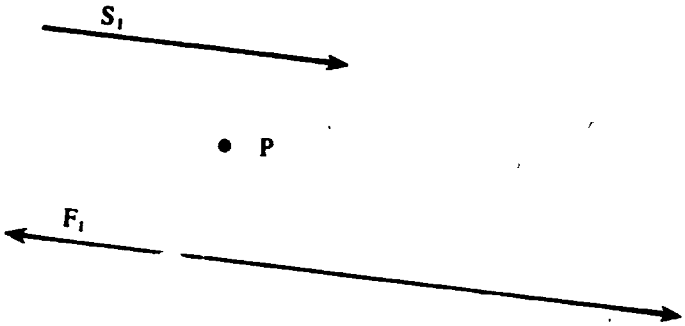
(b) If we let F_1 and F_2 be two flip lines which are not parallel, find $F_1F_2(P) = Q_1$, $F_2F_1(P) = Q_2$, $F_1F_2(R) = V_1$ and $F_2F_1(R) = V_2$ below.



Measure angles PIQ_1, PIQ_2, RIV_1, and RIV_2. What do you notice? Measure the "small" angle formed by the intersection of F_1 and F_2. Conjecture?

As in 2(a) above, we note that Q_1 and Q_2 are different points as are V_1 and V_2. Also F_1F_2 here looks like a turn with Center I as does F_2F_1. Try this with other lines and points and see what happens.

(c) If we let F_1 be a flip line and S_1 be a slide arrow parallel to F_1 below, find $S_1F_1(P) = Q_1$ and $F_1S_1(P) = Q_2$.



Here we again find that $Q_1 = Q_2$, and perhaps S_1F_1 looks like another slide. To convince yourself otherwise, pick a point R "below" line F_1 and find $S_1F_1(R)$.

It is interesting to note that successive motions of slides, flips or turns will always give congruent pre-images and images. However, there may be times when successive motions may not be as easily recognized as the exercises which have been proposed thus far. Specifically, the above activities do ~~not~~ comprise the total package of successive motions. Among those left to be considered are successive motions of: (a) a flip and a slide where the slide arrow is not parallel to the flip line, (b) a slide and a turn, (c) a turn and a flip, and (d) a turn and a turn where the turns do not have the same center of turn. Also, successive motions are not limited to two motions, but any number of successive motions can be accommodated. We have, however, covered the successive turns we feel that middle school students could explore in some detail. For a look at student materials on this topic, consult the materials produced by the University of Illinois Committee on School Mathematics entitled *Motion Geometry* (Phillips and Zwoyer, Harper and Row, New York, 1969).

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This publication in NEA's Curriculum Series is written for middle school teachers without specialized backgrounds in geometry. It is arranged in three parts and includes suggested activities and exercises for students.

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Part III provides a bibliography of additional sources for readings and activities in the area of geometry.

Geometric Selections for Middle School Teachers has been written by Douglas B. Aichele and Melfried Olson. Dr. Aichele is a Professor and Head of the Department of Curriculum and Instruction, Oklahoma State University, Stillwater. Dr. Olson is with the Science and Mathematics Teaching Center, University of Wyoming, Laramie.

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